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The complexity of tropical graph homomorphisms

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ABSTRACT

A tropical graph (H, c) consists of a graph H and a (not necessarily proper) vertex-colouring c of H. Given two tropical graphs (G, c_1) and (H, c), a homomorphism of (G, c_1) to (H, c) is a standard graph homomorphism of G to H that also preserves the vertex-colours. We initiate the study of the computational complexity of tropical graph homomorphism problems. We consider two settings. First, when the tropical graph (H, c) is fixed; this is a problem called (H, c)-Colouring. Second, when the colouring of H is part of the input; the associated decision problem is called H-Tropical-Colouring. Each (H, c)-Colouring problem is a constraint satisfaction problem (CSP), and we show that a complexity dichotomy for the class of (H, c)-Colouring problems holds if and only if the Feder–Vardi Dichotomy Conjecture for CSPs is true. This implies that (H, c)-Colouring problems form a rich class of decision problems. On the other hand, we were successful in classifying the complexity of at least certain classes of H-Tropical-Colouring problems.

1. Introduction

Unless stated otherwise, the graphs considered in this paper are simple, loopless and finite. A *homomorphism* h of a graph G to a graph H is a mapping $h : V(G) \to V(H)$ such that adjacency is preserved by h, that is, the images of two adjacent vertices of G must be adjacent in H. If such a mapping exists, we note $G \to H$. For a fixed graph H, given an input graph G, the decision problem H-Colouring (whose name is derived from the proximity of the problem to proper vertex-colouring) consists of determining whether $G \to H$ holds. Problems of the form H-Colouring for some fixed graph H, are called *homomorphism problems*. A classic theorem of Hell and Nešetřil [22] states a *dichotomy* for this problem: if H is bipartite, H-Colouring is polynomial-time solvable; otherwise, it is NP-complete.

Tropical graphs. As an extension of graph homomorphisms, homomorphisms of edge-coloured graphs have been studied, see for example [1,6–9]. In this paper, we consider the variant where the *vertices* are coloured. We initiate the study of *tropical graph homomorphism problems*, in which the vertex sets of the graphs are partitioned into colour classes. Formally, a *tropical graph* (G, c) is a graph G together with a (not necessarily proper) vertex-colouring $c : V(G) \to C$ of G, where C is a set of colours. If $|C| = k$, we say that (G, c) is a *k-tropical graph*. Given two tropical graphs (G, c_1) and (H, c_2) (where the colour set of c_1 is a subset of the colour set of c_2), a homomorphism h of (G, c_1) to (H, c_2) is a homomorphism of G to

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H that also preserves the colours, that is, for each vertex v of G, $c_1(v) = c_2(h(v))$. For a fixed tropical graph (H, c), problem (H, c)-Colouring asks whether, given an input tropical graph (G, c_1), we have $(G, c_1) \to (H, c)$.

The homomorphism factoring problem. Brewster and MacGillivray defined the following related problem in [10]. For two fixed graphs H and Y and a homomorphism h of H to Y, the (H, h, Y)-Factoring problem takes as an input, a graph G together with a homomorphism g of G to Y, and asks for the existence of a homomorphism f of G to H such that $f = h \circ g$. The (H, c)-Colouring problem corresponds to $(H, c, K^+_{|C|})$-Factoring where $K^+_{|C|}$ is the complete graph on $|C|$ vertices with all loops (and with C the set of colours used by c). (Note that in [10], loops were not considered.)

Constraint satisfaction problems (CSPs). Graph homomorphism problems fall into a more general class of decision problems, the *constraint satisfaction problems*, defined for *relational structures*. A relational structure S over a *vocabulary* (a vocabulary is a set of pairs (R_i, a_i) of relation names and arities) consists of a *domain* $V(S)$ of vertices together with a set of relations corresponding to the vocabulary, that is, $R_i \subseteq V(S)^{a_i}$ for each relation R_i of the vocabulary. Given two relational structures S and T over the same vocabulary, a homomorphism of S to T is a mapping $h : V(S) \to V(T)$ such that each relation R_i is preserved, that is, for each subset of $V(S)^{a_i}$ of R_i in S, its image set in T also belongs to R_i. For a fixed relational structure T, T-CSP is the decision problem asking whether a given input relational structure has a homomorphism to T.

Using this terminology, a graph H is a relational structure over the vocabulary $\{(A, 2)\}$ consisting of a single binary relation A (adjacency). Hence, H-Colouring is a CSP. Further, (H, c)-Colouring is equivalent to the problem $C(H, c)$-CSP, where $C(H, c)$ is obtained from H by adding a set of k unary relations to H (one for each colour class of the k-colouring c).

The Dichotomy Conjecture. In their celebrated paper [20], Feder and Vardi posed the following conjecture.

Conjecture 1.1 (*Feder and Vardi [20]*). *For every fixed relational structure T, T-CSP is polynomial-time solvable or NP-complete.*

Conjecture 1.1 became known as the *Dichotomy Conjecture* and has given rise to extensive work in this area, see for example [11,12,15–18]. If the conjecture holds, it would imply a fundamental distinction between CSP and the whole class NP. Indeed, the latter is known (unless P=NP) to contain so-called *NP-intermediate* problems that are neither NP-complete nor polynomial-time solvable [27].

The Dichotomy Conjecture was motivated by several earlier dichotomy theorems for special cases, such as the one of Schaefer for binary structures [29] or the one of Hell and Nešetřil for undirected graphs, stated as follows.

Theorem 1.2 (*Hell and Nešetřil Dichotomy [22]*). *Let H be an undirected graph. If H is bipartite, then H-Colouring is polynomial-time solvable. Otherwise, H-Colouring is NP-complete.*

Digraph homomorphisms. Digraph homomorphisms are also well-studied in the context of complexity dichotomies. We will relate them to tropical graph homomorphisms. For a digraph D, D-Colouring asks whether an input digraph admits a homomorphism to D, that is, a homomorphism of the underlying undirected graphs that also preserves the orientation of the arcs.

While in the case of undirected graphs, the H-Colouring problem is only polynomial time for graphs whose core is either K_1 or K_2, in the case of digraphs the problem remains polynomial time for a large class of digraphs which are cores. The classification of such cores has been one of the difficulties of the conjecture. Such classifications are given for certain interesting subclasses, see for example [2–5,14]. A proof of a conjecture in classification of general case has been announced while this paper was under review (see [19]). This would complete a proof of the Dichotomy Conjecture as Feder and Vardi [20] showed the following (seemingly weaker) statement to be equivalent to the Dichotomy Conjecture.

Conjecture 1.3 (*Equivalent Form of the Dichotomy Conjecture, Feder and Vardi [20]*). *For every bipartite digraph D, D-Colouring is polynomial-time solvable or NP-complete.*

In Section 3, similarly to its above reformulation (Conjecture 1.3), we will show that the Dichotomy Conjecture has an equivalent formulation as a dichotomy for tropical homomorphisms problems. More precisely, we will show that the Dichotomy Conjecture is true if and only if its restriction to (H, c)-Colouring problems, where (H, c) is a 2-tropical bipartite graph, also holds. In other words, one can say that the class of 2-tropical bipartite graph homomorphisms is as rich as the whole class of CSPs.

For many digraphs D it is known such that D-Colouring is NP-complete. Such a digraph of order 4 and size 5 is presented in the book by Hell and Nešetřil [23, page 151]. Such oriented trees are also known, see [24] or [23, page 158]; the smallest such known tree has order 45. A full dichotomy is known for oriented cycles [14]; the smallest such NP-complete oriented cycle has order between 24 and 36 [13,14]. Using these results, one can easily exhibit some NP-complete (H, c)-Colouring problems. To this end, given a digraph D, we construct the 3-tropical graph $T(D)$ as follows. Start with the set of vertices $V(D)$ and colour its vertices Blue. For each arc $\overrightarrow{uv}$ in D, add a path ux_ux_vv of length 3 from u to v in $T(D)$, where x_u and x_v are two new vertices coloured Red and Green, respectively. The following fact is not difficult to observe.

Proposition 1.4. *For any two digraphs D_1 and D_2, we have $D_1 \to D_2$ if and only if $T(D_1) \to T(D_2)$.*

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