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Communication

Vertex-colouring of 3-chromatic circulant graphs

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ABSTRACT

A circulant graph $C_n(a_1, \dots, a_k)$ is a graph with n vertices $\{v_0, \dots, v_{n-1}\}$ such that each vertex v_i is adjacent to vertices $v_{(i+a_j) \bmod n}$, for $j = 1, \dots, k$. In this paper we investigate the vertex colouring problem on circulant graphs. We approach the problem in a purely combinatorial way based on an array representation and propose an exact $O(k^3 \log^2 n + n)$ algorithm for a subclass of 3-chromatic $C_n(a_1, \dots, a_k)$'s with $k \geq 2$, which are characterized in the paper.

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1. Introduction

Consider $k+1$ integers $n, a_1, \dots, a_k$ such that $n > 0$, $k \geq 2$, $a_i \bmod n \neq 0$ for $i = 1, \dots, k$, and $a_i \not\equiv \pm a_j \pmod{n}$ for all $i = 1, \dots, k$, $j = 1, \dots, k$, $i \neq j$. The *circulant graph* $C_n(a_1, \dots, a_k)$ is the (simple undirected) graph with vertex set $\{v_0, v_1, \dots, v_{n-1}\}$ and edge set $\{(v_i, v_{(i+a_j) \bmod n}),\ \text{for } i = 0, \dots, n-1, j = 1, \dots, k\}$ (see Fig. 1). The integers $a_1, \dots, a_k$ are called *entries*. For simplicity reasons, in the sequel, a_1 and a_2 are often denoted a and b, respectively.

A circulant graph $C_n(a_1, \dots, a_k)$ is always regular: it is $2k$-regular if $a_i \neq \frac{n}{2}$ for all i, and $(2k-1)$-regular, otherwise. In addition, its adjacency matrix is symmetric and circulant.

Throughout the paper we assume w.l.o.g. that $a_i \in \{1, \dots, n-1\}$ for $i = 1, \dots, k$: under this hypothesis, the condition $a_i \not\equiv \pm a_j \pmod{n}$ is equivalent to require $a_i \neq a_j$ and $a_i + a_j \neq n$. These conditions imply $n \geq 2k$.

Consider an arbitrary entry $a_t \in \{a_1, \dots, a_k\}$. If $(x \pm a_t) \bmod n = y$, we say that $v_x, v_y \in V$ are a_t-*adjacent* and that (v_x, v_y) is an a_t-*edge*. By a_t-path and a_t-cycle we denote a path and a cycle, respectively, made of a_t-edges only. We observe that a circulant graph $C_n(a_1, \dots, a_k)$ contains $\gcd(n, a_t)$ distinct a_t-cycles with $\frac{n}{\gcd(n, a_t)}$ vertices each.

In this paper we investigate the vertex colouring problem on circulant graphs, which consists in finding a feasible k-colouring which minimizes k, where a *feasible k-colouring* is an assignment of k colours to the vertices of the given graph in such a way that adjacent vertices receive different colours. A graph G which admits a feasible k-colouring is k-*colourable*; the smallest k such that G is k-colourable is the *chromatic number* $\chi(G)$, and G is said to be $\chi(G)$-*chromatic*.

The vertex colouring problem on arbitrary graphs is known to be $\mathcal{NP}$-hard [20]. In [13], the vertex colouring problem on circulant graphs $C_n(a_1, \dots, a_k)$ is proved to be $\mathcal{NP}$-hard and not approximable better than a certain factor.

Related to the chromatic number $\chi(G)$ we would like to recall the *Lovász number* $\vartheta(G)$ of a graph G, which is an upper bound on the Shannon capacity of a graph, a well-known parameter in information theory. The so-called Sandwich Theorem states that $\omega(G) \leq \vartheta(\overline{G}) \leq \chi(G)$, where $\omega(G)$ denotes the clique number of a graph G, and $\overline{G}$ denotes the complement of G [6,8]. It is important to observe that the ϑ-function of a graph G can always be computed in polynomial time while the computation of $\omega(G)$ and $\chi(G)$ is $\mathcal{NP}$-hard.

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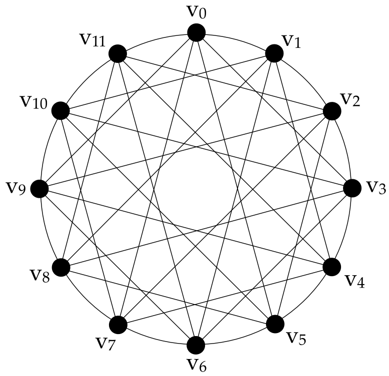

Fig. 1. The circulant graph $C_{12}(1, 3, 5)$, with $k = 3$ entries.

The computational complexity of the problem suggests to focus on circulant graphs with a fixed number k of entries.

For $k = 2$ the problem is studied in [18,19,27], where it is shown that it can be solved in polynomial time. In particular, in [18] the authors propose colouring algorithms for some 3-chromatic $C_n(1, b)$'s. In [27] the authors prove a conjecture by [14] which characterizes all the 3-chromatic $C_n(1, b)$'s. In [24] the existence of an n_0 such that $\chi(C_n(a, b)) \leq 3$ when $n \geq n_0$, $1 \leq a < b$, and $b \neq 2a$ is proved. In [19] optimal colouring algorithms are proposed for all $C_n(a, b)$'s: many different subcases are identified, depending on the value of some parameters, a few of which related to the Hermite Normal Form associated to the given $C_n(a, b)$, and an optimal colouring algorithm is proposed for each subcase. The following theorem fully characterizes the chromatic number of a circulant graph with two entries.

Theorem 1.1 (*[18,19,27]*)**.** *Consider a circulant graph $C_n(a, b)$ with $\gcd(n, a, b) = 1$ and $\gcd(n, a) \leq \gcd(n, b)$. Then*

$$\chi(C_n(a,b)) = \begin{cases} 2, & \text{if } n \text{ even}, \ a, b \text{ odd} \\ 5, & \text{if } n = 5 \\ 4, & \text{if } n = 13 \text{ AND } \big(b \equiv \pm 5a (\text{mod } 13) \text{ OR } a \equiv \pm 5b (\text{mod } 13)\big) \\ 4, & \text{if } n \neq 5, n \bmod 3 \neq 0, \text{ AND } \big(b \equiv \pm 2a \ (\text{mod } n) \text{ OR } a \equiv \pm 2b \ (\text{mod } n)\big) \\ 3, & otherwise. \end{cases}$$

When $k = 3$ some partial results are known: the authors of [27] characterize 4-chromatic 6-regular $C_n(1, b, b+1)$'s, proving a conjecture by [14]; the chromatic number of $C_n(a, b, a+b)$'s is studied in [21], where optimal colouring algorithms are also proposed; in [2] the chromatic number of all circulant graphs $C_n(a_1, a_2, a_3)$ with $a_1 < a_2 < a_3$ and $n \geq 4a_2a_3$ is computed: the results are summarized in their Theorem 6, which shows that most of those graphs are 3-chromatic, except some particular cases which are 4- or 5-chromatic. The method by [19] for colouring circulant graphs $C_n(a_1, a_2)$ is extended in [22], resulting in the characterization of the chromatic number of 5-regular $C_n(a_1, a_2, \frac{n}{2})$'s and in optimal colouring algorithms for such graphs. However, as pointed out in [22], the "proof is fairly complicated" and "it does not appear feasible to determine the chromatic number of 6-valent circulants" $C_n(a_1, a_2, a_3)$'s by generalizing the method of [19]. Other results are found in [1,5,6,8,9,25].

No matter about the number k of entries, a circulant graph $C_n(a_1, \ldots, a_k)$ is bipartite (thus 2-chromatic) if and only if n is even and all the entries are odd. The authors in [13], using spectral techniques, show that $\lceil \log p \rceil$ eigenvectors are necessary and sufficient to feasibly colour p-chromatic circulant graphs of degree less than 5, and show how to use the signs of the eigenvectors to colour the graph. In [15] two heuristic algorithms are proposed to find the set of eigenvectors which gives the best colouring. The authors of [24] prove the existence of an n_0 such that $\chi(C_n(a_1, \ldots, a_k)) \leq 3$ when $n \geq n_0$ and $a_1 < \cdots < < a_k < 2a_1$. In [2] the authors discuss some upper bounds on the chromatic number of circulant graphs whose set of entries has particular properties. Finally, the author of [29] states that upper bounds for the circular chromatic number and the chromatic number of circulant graphs can be obtained by means of the regular colouring method proposed in that paper for distance graphs (a *distance graph* $G_{\mathbb{Z}}(a_1, a_2, \ldots, a_k)$ is a graph with an infinite number of vertices $\{\ldots, v_{-2}, v_{-1}, v_0, v_1, v_2, \ldots\}$, where two vertices v_x and v_y are adjacent if and only if $|x-y| \in \{a_1, a_2, \ldots, a_k\}$, see [10,11,16]).

To our knowledge, no characterization is known for the chromatic number of circulant graphs $C_n(a_1, \ldots, a_k)$ with $k \geq 4$.

In this paper, we propose a purely combinatorial approach for the vertex-colouring problem on circulant graphs. We consider the circulant graphs $C_n(a_1, \ldots, a_k)$'s with two entries a_i, a_j verifying $\gcd(n, a_i, a_j) = 1$. Under this hypothesis, any other entry a_t can be expressed as a weighted combination of a_i and a_j. We prove that the non-bipartite circulant graphs where, for each entry, the sum of the weights is odd and the weights themselves undergo some constraints are 3-chromatic. We also discuss an exact colouring algorithm for them.

The paper is organized as follows. Preliminary definitions and results can be found in Section 2. In Section 3 we define the subclass $\mathcal{T}$ of circulant graphs admitting a certain structure, the *staircase*, having prescribed properties. We also propose

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