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Note

The number of spanning trees of a family of 2-separable weighted graphs

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ABSTRACT

Based on electrically equivalent transformations on weighted graphs, in this paper, we present a formula for computing the number of spanning trees of a family of 2-separable graphs formed from two base graphs by 2-sum operations. As applications, we compute the number of spanning trees of some special 2-separable graphs. Then comparisons are made between the number of spanning trees and the number of acyclic orientations for this family of 2-separable graphs under certain constraints. We also show that a factorization formula exists for the sum of weights of spanning trees of a special splitting graph.

1. Introduction

Throughout this paper, we only consider connected graphs that may have multi-edges but no loops. The path, the cycle, the complete graph of order n is denoted by P_n, C_n, K_n, respectively. Let $G = (V, E)$ be a graph with an edge-weight function $w : E \to [0, \infty)$, and assume that $|V| = n$ and $|E| = m$ if not specified. We use $\mathcal{T}(G)$ to denote the set of all spanning trees of the underlying graph G and $t(G)$ to denote the number of spanning trees of G, i.e. $t(G) = |\mathcal{T}(G)|$. For $T \in \mathcal{T}(G)$, we denote by $w(T) = \prod_{e\in E(T)} w(e)$ the *weight* of T and by $\tau(G) = \sum_{T\in\mathcal{T}(G)} w(T)$ the *sum of weights of spanning trees* of G. Clearly, $\tau(G) = t(G)$ if $w(e) = 1$ for all $e \in E(G)$. The *Laplacian matrix* $L(G) = (l_{ij})_{n\times n}$ of G is defined by its entries

$$l_{ij} = -\sum_{\substack{e\in E\\ e \text{ connects } i,j}} w(e) \text{ and } l_{ii} = -\sum_{j\in V, j\neq i} l_{ij},$$

where $i, j \in V$, $i \neq j$. The well-known *Matrix Tree Theorem* (see e.g. [2]) expresses $\tau(G)$ as the value of a certain determinant, i.e. $\tau(G) = (-1)^{i+j}\det(L_i^j(G))$, where $L_j^i(G)$ is the submatrix obtained from $L(G)$ by deleting the ith row and the jth column.

A 2-separable graph [16] is the resulting graph $\hat{G}$ obtained from a graph G by replacing each edge $e = xy$ by a connected graph H_e (here H_e varies across the edge e) that has only the vertices x and y in common with the rest of $\hat{G}$. For the purpose of the present paper, we pay attention to a class of special 2-separable graphs as below. Let G be a graph and H be an edge-weighted graph with at least two distinct vertices u and v. The H-replacement of G, denoted by $\widehat{G[H]}$, is defined as the graph obtained by replacing each edge e of G by a copy of H, that is, deleting the edge e in G and identifying pairs of vertices $\{u, e_u\}$ and $\{v, e_v\}$, where e_u and e_v are ends of e. For instance, if $H = P_3$ and u, v are its two end vertices, then $\widehat{G[P_3]}$ is the subdivision of G; if $H = K_3$ and u, v are taken arbitrarily, then $\widehat{G[K_3]}$ is the triangulation of G that is obtained from G by

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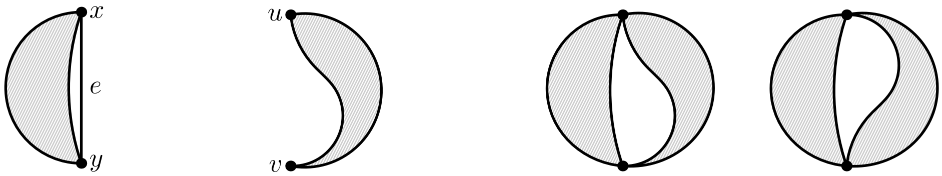

(a) G. (b) H. (c) Two types of replacements on e.

Fig. 1. Two resulting graphs by replacing the edge e in G by H.

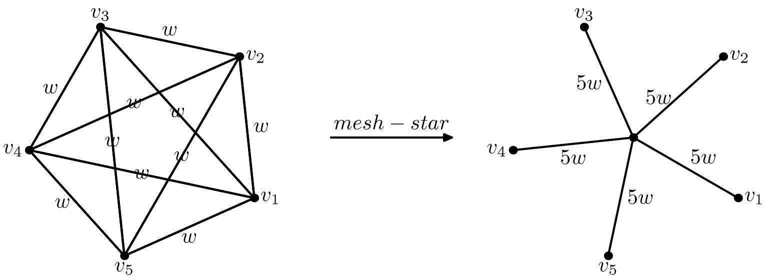

Fig. 2. The mesh-star transformation for $n = 5$.

adding a new vertex corresponding to each edge and connecting it to the ends of the edge considered. Each replacement is analogous to the so-called 2-sum operation [7], and it may not be unique since H can be reversed end-for-end (see Fig. 1), but two resulting graphs are 2-isomorphic [15] and thus have the same number of spanning trees. Thus there is no reason to distinguish between different replacements. The graphs resulting by replacements have received attention on several aspects including, but not limited to, the perfect matching [17], the matching polynomial [18], the number of spanning trees and spanning forests [9,12], the resistance distance [19] and the Tutte polynomial [5,16]. In this paper, we consider the relationship between $\tau(G)$, $\tau(H)$ and $\tau(\widehat{G[H]})$, and discuss applications.

2. Preliminaries

Following [9,11,12], we recall some concepts, notations and results from electrical network theory. An electrical network can be regarded as an edge-weighted graph in which the weights are the conductances of the respective edges. Two edge-weighted graphs (networks) G and H are called *electrically equivalent* with respect to $D \subseteq V(G) \cap V(H)$, if they cannot be distinguished by applying voltages to D and measuring the resulting currents on D. If $u, v \in V(G)$ and $H = K_2$ with vertex set $\{u, v\}$, then there exists an *effective conductance* $w_{\text{eff}}(u, v)$ on the single edge of H, so that G and H are electrically equivalent with respect to $\{u, v\}$. The *effective resistance* between u and v is denoted by $r_{\text{eff}}(u, v) = w_{\text{eff}}(u, v)^{-1}$. Let $u, v \in V(G)\,(u \neq v)$. Then a rooted forest with two roots u, v, also known as a *thicket* [1], is a spanning forest with exactly two components such that u and v belong to different components. By all minors matrix-tree theorem [3], the number of thickets of G with roots u, v is given by $\det(L_{uv}^{uv}(G))$, where $L_{uv}^{uv}(G)$ be the submatrix obtained from $L(G)$ by deleting the rows and columns corresponding to u, v.

Lemma 2.1 (*[1] Theorem 1 or [10] as a special case of Theorem 2*).

$$r_{\text{eff}}(u, v) = \frac{\det(L_{uv}^{uv}(G))}{\tau(G)}.$$

In [11], Teufl and Wagner explored the effect of electrically equivalent transformations on the sum of weights of spanning trees.

Theorem 2.2 (*[11] Theorem 10*). *Let X be a weighted graph that can be partitioned into two edge-disjoint subgraphs G and H (inheriting weights in an obvious way) such that $V(X) = V(G) \cup V(H)$, $V(G) \cap V(H) = S$. We assume that $\tau(X) \neq 0$ and $\tau(H) \neq 0$. Let H' be a weighted graph satisfying $E(G) \cap E(H') = \emptyset$ and $V(G) \cap V(H') = S$. Furthermore, suppose H and H' are electrically equivalent with respect to S. Finally, set $X' = G \cup H'$. Then, the following formula holds:*

$$\frac{\tau(X')}{\tau(X)} = \frac{\tau(H')}{\tau(H)}. \tag{2.1}$$