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Completing colored graphs to meet a target property[☆]

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ABSTRACT

We consider the problem of deciding whether a k-colored graph can be completed to have a given property. We establish that, when k is not fixed, the completion problems for (Helly) circular-arc graphs, even (Helly) proper or (Helly) unit circular-arc graphs, are NP-complete. When k is fixed, in the case of completion to a (Helly) circular-arc graph, (Helly) proper circular-arc graph, or (Helly) unit circular-arc graph we fully classify the complexities of the problems. We also show that deciding whether a 3-colored graph can be completed to be strongly chordal can be done in $O(n^2)$ time. As a corollary of our results, the sandwich problems for Helly circular-arc graphs, Helly proper circular-arc graphs, and Helly unit circular-arc graphs are NP-complete.

1. Introduction

The graphs that we consider are simple and any vertex coloring considered is proper. A *colored graph* is a graph properly colored with an arbitrary number of colors. A *k-colored graph* is a graph properly colored with k colors. In the Π sandwich problem, given graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ on the same vertex-set with $E_1 \subseteq E_2$ and a property Π, the question is whether there is a graph $G = (V, E)$ that has property Π and also $E_1 \subseteq E \subseteq E_2$ holds. The set E_1 contains the *required edges* while the set $E_2 \setminus E_1$ contains the *optional edges*. It is seen that when $E_1 = E_2$ the sandwich problem is the same as the recognition problem for property Π. The sandwich problem was introduced in 1995 by Golumbic et al. in [14] where they studied the problem for the property of membership in several classes of perfect graphs. Since then, the sandwich problem has been studied for a variety of NP properties, and a number of published papers can be found on the topic.

In the Π *completion of a colored graph* problem, given a property Π and a graph $G = (V, E)$ with a proper vertex coloring $c : V \to Z$, the question is whether there exists a supergraph $G' = (V, E')$ of G that has property Π and also is properly colored by c. When such a G' exists, we say G' is a Π *completion of* G and G *admits a* Π *completion*. By taking the edges in E to be the required edges and $\{xy \mid x \in V, y \in V, xy \notin E, \text{ and } c(x) \neq c(y)\}$ to be the set of optional edges, it is seen that the Π completion of a colored graph is a restriction of the sandwich problem for property Π. There are sandwich problems that are NP-complete (such as for comparability graphs), whose corresponding colored graph completion problem is trivial (as every complete k-partite graph is a comparability graph). However, there are also NP-complete sandwich problems whose colored graph completion version remains hard. Π completion of colored graphs has been studied inside and outside the context of the sandwich problem.

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It was established in [28] that the perfect phylogeny problem is equivalent to the problem of chordal completion of colored graphs. It was subsequently shown that this problem is NP-complete [4,5], which in turn implies that the sandwich problem for chordal graphs is also NP-complete. The sandwich problems for strongly chordal graphs and chordal bipartite graphs were two of the problems whose complexities were left as an open question in [14]. It was shown in [7,24] that these problems are NP-complete; these proofs established that the problems of strongly chordal completion of colored graphs and chordal bipartite completion of colored graphs are NP-complete.

The complexity of chordal completion of a colored graph has been studied when the input graph is colored with k colors, where k is constant. For the case that $k = 3$, several linear-time algorithms are known [6,16,17]. It is also known that for every fixed k, the problem can be solved in polynomial time [22].

Due to potential applications to the problem of DNA physical mapping [3,13], the problem of interval completion of a colored graph, and its variations, have been studied extensively in the literature. It is known that the problems of interval completion of a colored graph [3] and unit interval completion of a colored graph [13] are NP-complete. Several results are known for interval completion and unit interval completion of a colored graph when the input is restricted. Let k be the number of colors used on the input graph. In contrast to the case for chordal graphs, the best known algorithm for the interval completion problem when $k = 3$ runs in $O(n^2)$ time [3]. In further contrast, the interval completion of a k-colored graph problem has been shown to be NP-complete for every fixed $k \geq 4$ [3]. In fact, the interval completion of a colored graph problem is NP-complete even for four-colored caterpillars [1]. Similarly, it is known that the unit interval completion of a colored graph is solvable in polynomial time for caterpillars with hair length less than 2 [2], while it is NP-complete for caterpillars with hair length at least 2 [2].

The main contributions of this paper are as follows: First, we provide simple proofs that, given a graph colored with an arbitrary number of colors, each of the problems (Helly) circular-arc completion, (Helly) proper circular-arc completion, and (Helly) unit circular-arc completion is NP-complete. Later we prove these problems remain NP-complete when the number of colors is fixed and at least 3. Then, we provide a full classification of the complexity of the problem of completing a colored graph to be a circular-arc graph when the number of colors used is fixed. Specifically, given a k-colored graph, we show that when $k = 2$, there is an $O(n)$-time algorithm for the problem, but when $k = 3$, the problem is NP-complete; in turn, the problem remains NP-complete for every fixed k, $k \geq 3$. We provide an identical classification for the problem of completing a colored graph to be a Helly circular-arc graph, proper circular-arc graph, unit circular-arc graph, Helly proper circular-arc graph, or Helly unit circular-arc graph. To the best of our knowledge these are the first instances of a colored graph completion problem on 3-colored graphs that are hard. We also show that deciding whether a 3-colored graph admits a strongly chordal completion can be decided in $O(n^2)$ time; our algorithm is based on a characterization of bi-connected 3-colored graphs that admit a strongly chordal completion. We conjecture that the corresponding problem for 4-colored graphs is NP-complete. It is known that the sandwich problems for interval graphs, circular-arc graphs, and proper (resp., unit) circular-arc graphs are NP-complete [14]. It follows from our results that the sandwich problem for Helly circular-arc graphs and the sandwich problem for Helly proper (resp., unit) circular-arc graphs are NP-complete.

2. Definitions

All graphs considered in this paper are finite, undirected, and simple (i.e., without loops and multiple edges). For notations and terminology not defined here, we refer to [29]. Let $G = (V, E)$ be a graph. We use n to refer to $|V|$ and m to refer to $|E|$. For $S \subseteq V$, $G[S]$ refers to the subgraph of G induced by S. For a vertex $v \in V$ the *(open) neighborhood* of v is the set $N(v) = \{u \in V \mid uv \in E\}$, while the *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. We use $N_G(v)$ and $N_G[v]$ to refer to an open (resp., closed) neighborhood in G if necessary for clarity. Vertex x is *simplicial* if $N(x)$ is a clique. The complete graph with k vertices is denoted by K_k. The complete bipartite graph $B(X, Y, E)$, where $|X| = 1$, $|Y| = 3$, and $E = \{xy \mid x \in X \text{and} y \in Y\}$, is denoted by $K_{1,3}$ (also known as the *claw*). A path with k vertices is denoted P_k.

G is *chordal* if every cycle with at least 4 vertices in G has a chord. For $k \geq 1$, a *k-tree* is defined recursively as follows: K_{k+1} is a k-tree. Given a k-tree T_n on n vertices, a k-tree T_{n+1} on $n+1$ vertices can be constructed by adding a new vertex whose neighborhood is a clique of size k in T_n. A *partial k-tree* is a subgraph of a k-tree. An *n-sun* is the graph on $2n$ vertices ($n \geq 3$) whose vertex set can be partitioned into $W = \{w_0, \ldots, w_{n-1}\}$ and $U = \{u_0, \ldots, u_{n-1}\}$ such that U is a clique, W is an independent set, and u_i is adjacent to w_j if and only if $i = j$ or $i = j + 1 \pmod n$. A graph is *strongly chordal* if it is chordal and does not contain an n-sun, $n \geq 3$.

G is an *interval graph* if every $x \in V$ can be mapped to an interval I_x on the real line such that $xy \in E$ if and only if $I_x \cap I_y \neq \emptyset$; $\{I_x \mid x \in V\}$ is an *interval model* for G. G is a *circular-arc graph* if every $x \in V$ can be mapped to an arc A_x on a circle such that $xy \in E$ if and only if $A_x \cap A_y \neq \emptyset$; $\{A_x \mid x \in V\}$ is a *circular-arc model* for G. A circular-arc graph G is a *proper circular-arc graph* if G has a circular-arc model in which no arc properly contains another arc. A circular-arc graph G is a *unit circular-arc graph* if G has a circular-arc model in which all arcs have unit length. *Proper interval graphs* and *unit interval graphs* are similarly defined. By definition, unit circular-arc graphs (resp., unit interval graphs) are a subclass of proper circular-arc graphs (resp., proper interval graphs). In circular-arc models and proper circular-arc models the arcs can be adjusted so that no two arcs share an endpoint. Thus, these classes remain the same whether arcs are open or closed. In the case of unit circular-arc graphs (resp., unit interval graphs), we assume the original definition for circular-arc models (resp., interval models) in which arcs (resp., intervals) are restricted to be all closed (or all open). Under this assumption, we have the following relationships between classes. Roberts [23] proved a graph is a unit interval graph if and only if it is a

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