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# Partitioning the vertices of a cubic graph into two total dominating sets

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ABSTRACT

A total dominating set in a graph $G$ is a set $S$ of vertices of $G$ such that every vertex in $G$ is adjacent to a vertex of $S$. We study cubic graphs whose vertex set can be partitioned into two total dominating sets. There are infinitely many examples of connected cubic graphs that do not have such a vertex partition. In this paper, we show that the class of claw-free cubic graphs has such a partition. For an integer $k$ at least 3, a graph is $k$-chordal if it does not have an induced cycle of length more than $k$. Chordal graphs coincide with 3-chordal graphs. We observe that for $k \geq 6$, not every graph in the class of $k$-chordal, connected, cubic graphs has two vertex disjoint total dominating sets. We prove that the vertex set of every 5-chordal, connected, cubic graph can be partitioned into two total dominating sets. As a consequence of this result, we observe that this property also holds for a connected, cubic graph that is chordal or 4-chordal. We also prove that cubic graphs containing a diamond as a subgraph can be partitioned into two total dominating sets.


## 1. Introduction

A *dominating set* in a graph $G$ is a set $S$ of vertices of $G$ such that every vertex in $V(G) \setminus S$ is adjacent to at least one vertex in $S$. A *total dominating set*, abbreviated as a TD-set, of a graph $G$ with no isolated vertex is a set $S$ of vertices of $G$ such that every vertex in $V(G)$ is adjacent to at least one vertex in $S$. The literature on the subject of domination parameters in graphs up to the year 1997 has been surveyed and detailed in the two books [9,10]. Total domination is now well studied in graph theory. For a recent book on the topic, see [15]. A survey of total domination in graphs can also be found in [11].

For an integer $k$ at least 3, a graph $G$ is *k-chordal* if it does not have an induced cycle of length more than $k$. Chordal graphs coincide with 3-chordal graphs, and graphs of small chordality were studied in [2,4,5,19]. We say that a graph is *F-free* if it does not contain $F$ as an induced subgraph. In particular, if $F = K_{1,3}$, then we say that the graph is *claw-free*. An excellent survey of claw-free graphs has been written by Flandrin, Faudree, and Ryjáček [8].

A simple yet fundamental observation in domination theory made by Ore [21] is that every graph of minimum degree at least one contains two disjoint dominating sets. Thus, the vertex set of every graph without isolated vertices can be partitioned into two dominating sets. In contrast to that, Zelinka [25,26] showed that no minimum degree is sufficient to guarantee the existence of two disjoint total dominating sets.

Our aim in this paper is to study 3-regular graphs, also called *cubic graphs*, whose vertex set can be partitioned into two total dominating sets. There are infinitely many examples of connected cubic graphs that do not have such a vertex partition. We show that the class of 5-chordal, cubic graphs and the class of claw-free, cubic graphs have such a partition.

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### 1.1. Notation

Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The *open neighborhood* of a vertex $v \in V(G)$ is $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$ and its *closed neighborhood* is the set $N_G[v] = \{v\} \cup N_G(v)$. Given a subset $S \subseteq V(G)$ and a vertex $v \in S$, the *S-external private neighborhood* of $v$ in $G$ is the set $\mathrm{epn}_G(v, S) = \{w \in V(G) \setminus S \mid N_G(w) \cap S = \{v\}\}$. The subgraph induced by a set of vertices $S$ in $G$ is denoted by $G[S]$. We use the standard notation $[k] = \{1, 2, \ldots, k\}$. The complete graph on four vertices minus one edge is called a *diamond*.

A *block* of a graph $G$ is a maximal connected subgraph of $G$ which has no cut-vertex of its own. The blocks of $G$ partition its edge set. If a connected graph contains a single block, we call the graph itself a *block* or a 2-*connected graph*. A block containing exactly one cut-vertex is called an *end-block*. We note that every graph that contains a cut-vertex contains an end-block.

Let $v$ be a vertex of a digraph $D$. The *outdegree* $d^+(v)$ of $v$ is the number of arcs leaving $v$; that is, $d^+(v)$ is the number of arcs of the form $(v, u)$. The *indegree* $d^-(v)$ of $v$ is the number of arcs entering $v$. The digraph $D$ is *k-regular* if $d^+(v) = d^-(v) = k$ for every vertex $v \in V$. By a *cycle* in $D$, we mean a directed cycle. An *even cycle* in $D$ is a cycle of even length.

A *hypergraph* $H$ is a finite set $V(H)$ of elements, called *vertices*, together with a finite multiset $E(H)$ of arbitrary subsets of $V$, called *hyperedges* or simply *edges*. The hypergraph $H$ is *k-uniform* if every edge of $H$ has size $k$. The *degree* of a vertex $v$ in $H$ is the number of edges of $H$ which contain $v$. We denote the maximum degree among the vertices of $H$ by $\Delta(H)$. The hypergraph $H$ is *k-regular* if every vertex has degree $k$ in $H$.

Two vertices $x$ and $y$ of a hypergraph $H$ are *adjacent* if there is an edge $e$ of $H$ such that $\{x, y\} \subseteq e$. Further, $x$ and $y$ are *connected* if there is a sequence $x = v_0, v_1, v_2, \ldots, v_k = y$ of vertices of $H$ in which $v_{i-1}$ is adjacent to $v_i$ for $i \in [k]$. A *connected hypergraph* is a hypergraph in which every pair of vertices is connected. A *component* of a hypergraph $H$ is a maximal connected subhypergraph of $H$.

The *open neighborhood hypergraph* $\mathrm{ONH}(G)$ of a graph $G$ is the hypergraph whose vertex set is $V(G)$ and whose edges consist of the open neighborhoods of the vertices in $G$. Thus, if $H = \mathrm{ONH}(G)$, then $V(H) = V(G)$ and $E(H) = \{N_G(x) \mid x \in V(G)\}$.

A subset $T$ of vertices in a hypergraph $H$ is a *transversal* in $H$ if $T$ has a nonempty intersection with every edge of $H$. The *transversal number* $\tau(H)$ of $H$ is the minimum size of a transversal in $H$. It is well-known (see [15]) that the transversal number of the open neighborhood hypergraph of a graph is precisely the total domination number of the graph; that is, for a graph $G$, we have $\gamma_t(G) = \tau(H_G)$.

We define a partition of the vertex set of a graph into two disjoint TD-sets to be a *TDTD-partition* of the graph. If $G$ has a TDTD-partition, we say that $G$ is a *TDTD-graph*.

## 2. Main results

In this paper, we prove three main results. Our first result provides a local property for a connected, cubic graph to be a TDTD-graph. A proof of Theorem 1 is presented in Section 3.

**Theorem 1.** *Every connected cubic graph containing a diamond has a TDTD-partition.*

Our next two results provide global properties for a connected, cubic graph to be a TDTD-graph. The following result shows that the class of 5-chordal cubic graphs are TDTD-graphs. We show later (see Observation 8) that for $k \geq 6$, not every graph in the class of connected, $k$-chordal, cubic graphs is a TDTD-graph. A proof of Theorem 2 is presented in Section 4.

**Theorem 2.** *Every connected,* 5*-chordal, cubic graph has a TDTD-partition.*

Our third result shows that the class of claw-free, cubic graphs are TDTD-graphs. A proof of Theorem 3 is presented in Section 5.

**Theorem 3.** *Every connected, claw-free, cubic graph has a TDTD-partition.*

### 2.1. Motivation

A hypergraph $H$ is *bipartite* if its vertex set can be partitioned into two sets such that every hyperedge intersects both partite sets. Equivalently, $H$ is bipartite if it is 2-*colorable*; that is, there is a 2-coloring of the vertices such that each hyperedge contains two vertices of distinct colors. In other words, there must be no monochromatic hyperedge. The problem of 2-colorings of hypergraphs has attracted much interest over the past few decades, including important contributions by Seymour [22] and Thomassen [24]. Recent results on 2-colorings of hypergraphs can be found in [16,18] and elsewhere. We remark that the Lovász Local Lemma was devised by Erdös and Lovász in 1975 (see [7]) precisely to tackle the problem of 2-colorings of hypergraphs. There is a surprising connection between disjoint total dominating sets in graphs, even cycles in digraphs, and 2-coloring of hypergraphs.