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Choosability and paintability of the lexicographic product of graphs

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ABSTRACT

This paper studies the choice number and paint number of the lexicographic product of graphs. We prove that if G has maximum degree Δ, then for any graph H on n vertices $\text{ch}(G[H]) \le (4\Delta + 2)(\text{ch}(H) + \log_2 n)$ and $\chi_P(G[H]) \le (4\Delta + 2)(\chi_P(H) + \log_2 n)$.

1. Introduction

A *list assignment* L of a graph G assigns to each vertex v a set $L(v)$ of permissible colors. An L-*coloring* of G is a proper vertex coloring of G which colors each vertex with one of its permissible colors. We say that G is L-*colorable* if there exists an L-coloring of G. Given a function $f : V(G) \to \mathbb{N}$, we say G is f-*choosable* if for every list assignment L with $|L(v)| = f(v)$ for all $v \in V(G)$, G is L-colorable. We say G is k-choosable if for the constant function $f \equiv k$, G is f-choosable. The *choice number* of G is $\text{ch}(G) = \min\{k : G \text{ is } k\text{-choosable}\}$. List coloring of graphs has been studied extensively in the literature (cf. [12,6,11]).

More generally, we say that G is (a, b)-*choosable* for some integers a and b, $a \ge 2b > 1$, if, for any assignment of lists with $|L(v)| = a$ for all $v \in V$, there are subsets $C(v) \subset L(v)$ with $|C(v)| = b$ such that $C(u)$ and $C(v)$ are disjoint for all pairs of adjacent vertices u and v (the sets $C(v)$ form a b-*fold* L-*coloring*). The b-choice number of a graph is $\text{ch}_b(G) = \min\{a : G \text{ is } (a, b)\text{-choosable}\}$.

Motivated by practical applications and by the emphasis that computer science puts on algorithmic aspects, investigation of online variants of combinatorial problems has attracted considerable attention. Besides having direct implications, online variants usually also give a deeper insight into the original offline problem. As for many other coloring problems, online variant of list coloring came into the attention of researchers.

Assume that $\cup_{v\in V(G)} L(v) = \{1, 2, \ldots, q\}$ for some integer q. For $i = 1, 2, \ldots, q$, let $V_i = \{v : i \in L(v)\}$. The sequence $(V_1, V_2, \ldots, V_q)$ is another way of specifying the list assignment. An L-coloring of G is equivalent to a sequence $(X_1, X_2, \ldots, X_q)$ of independent sets that form a partition of $V(G)$ and such that $X_i \subseteq V_i$ for $i = 1, 2, \ldots, q$. This point of

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view of list coloring motivates the definition of online (b-fold) list coloring in the form of the following painting game on a graph G, which was introduced in [9,10].

Definition 1.1. Given a finite graph G and two mappings $f, g : V(G) \to \mathbb{N}$, the g-fold f-painting game on G is played by two players, Lister and Painter. In the ith round, Lister presents a non-empty subset V_i of $V(G)$, where each $v \in V_i$ is contained in less than $g(v)$ of the X_j's for $j \leq i-1$, and Painter chooses an independent set X_i contained in V_i. If at the end of some round, a vertex v is contained in $f(v)$ of the sets V_i's but contained in less than $g(v)$ of the sets X_i's, then the game ends and Lister wins the game. Otherwise, at some round, each vertex v is contained in $g(v)$ of the sets X_i, the game ends and Painter wins the game.

If $v \in V_i$, then we say color i is a permissible color of vertex v. If $v \in X_i$, then we say v is colored by color i.

We say G is *g-fold f-paintable* if Painter has a winning strategy in the g-fold f-painting game on G, and we say G is *b-fold k-paintable* if G is g-fold f-paintable for the constant functions $f \equiv k$ and $g \equiv b$. The *b-paint number* of G, denoted by $\chi_{P,b}(G)$, is the least integer k such that G is b-fold k-paintable. The *paint number* of G, denoted by $\chi_P(G)$, is defined as $\chi_P(G) = \chi_{P,1}(G)$, that is, it is the least integer k such that G is 1-fold k-paintable.

Many papers studied the connection of this online list coloring with the usual list coloring. First, it follows from the definition that for any graph G, $\chi_P(G) \geq \mathrm{ch}(G)$ and $\mathrm{ch}_b(G) \leq \chi_{P,b}(G)$. It was proved in [5] that the difference $\chi_P(G) - \mathrm{ch}(G)$ can be arbitrarily large for complete bipartite graphs. On the other hand, many currently known upper bounds for the choice numbers of classes of graphs remain upper bounds for their paint number. For example, the paint number of planar graphs is at most 5 [9], the paint number of planar graphs of girth at least 5 is at most 3 [9,4], the paint number of the line graph $L(G)$ of a bipartite graph G is $\Delta(G)$ [9], the maximum degree of G, and if G has an orientation in which the number of even Eulerian[1] subgraphs differs from the number of odd Eulerian subgraphs and $f(x) = d^+(x) + 1$, then G is f-paintable [10].

To avoid confusion we mention that a different list coloring game, *game list coloring*, was investigated by Borowiecki, Sidorowicz and Tuza [3]. In their list coloring game, the graph and the lists are given at the beginning of the game, and both players color vertices with colors from their lists, however, with opposite goals.

Various graph products, such as direct product, Cartesian product, strong product and lexicographic product, are important and popular methods of constructing new graphs from old ones. It is interesting to understand how a graph parameter of a product graph relates to graph parameters of the factor graphs. In this paper, we are interested in the choice number and the paint number of the lexicographic product of graphs.

Definition 1.2. Let $G = (V_1, E_1)$ and $H = (V_2, E_2)$ be two graphs. The *lexicographic product* of G and H is the graph $G[H]$ with vertex set $V_1 \times V_2$ and (v_1, v_2) is joined to (v_1', v_2') if either $(v_1, v_1') \in E_1$ or $v_1 = v_1'$ and $(v_2, v_2') \in E_2$.

Lexicographic products have a close connection with fractional colorings. The *fractional chromatic number* of a graph G is defined as $\chi_F(G) = \inf \chi(G[K_n])/n$, where the infimum is over all n. The *fractional choice number* or *choice ratio* is defined as $\mathrm{ch}_F(G) = \inf\{\mathrm{ch}_b(G)/b\} = \inf\{a/b : G \text{ is } (a,b)\text{-choosable}\}$. In the same vein, *fractional paint number* is defined as $\chi_{P,F}(G) = \inf\{\chi_{P,b}(G)/b\}$. Note that the b-choice number can be imagined as a restricted version of the choice number of the lexicographic product $G[K_b]$, in which the lists are restricted such that all vertices of a copy of K_b have the same set of permissible colors. Thus $\mathrm{ch}_b(G) \leq \mathrm{ch}(G[K_b])$. Similarly, $\chi_{P,b}(G) \leq \chi_P(G[K_b])$.

It follows from the definition that $\chi_F(G) \leq \mathrm{ch}_F(G)$ for all graphs G. Although $\mathrm{ch}(G) - \chi(G)$ can be arbitrarily large, Alon, Tuza and Voigt [2] showed the equality $\mathrm{ch}_F(G) = \chi_F(G)$ holds for all graphs G. This result was further strengthened by Gutowski [8], who showed that for any graph G, its fractional paint number also equals the fractional chromatic number. About the choice number of lexicographic products our knowledge is much more limited. In this paper we investigate the choice number of lexicographic products along with its online variant, the paint number of lexicographic products.

Throughout the paper $\log = \log_2$ stands for the base 2 logarithm and ln stands for the natural logarithm.

1.1. Our results and discussion

First we summarize the trivial relations between the coloring numbers we are interested in:

Observation 1.3. $\chi(G[H]) \leq \mathrm{ch}_{\chi(H)}(G) \leq \mathrm{ch}(G[H]) \leq \chi_P(G[H])$.

Our aim is to bound these coloring numbers with a function of parameters depending on G and H. For the chromatic number of lexicographic products, the following is trivially true:

Observation 1.4. $\chi(G[H]) \leq \chi(G)\chi(H)$.

[1] A spanning subgraph of a directed graph is even (resp. odd) Eulerian if it has an even (resp. odd) number of edges, and for each vertex its in-degree equals its out-degree.

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