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Distance and distance signless Laplacian spread of connected graphs☆

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ABSTRACT

For a connected graph G on n vertices, recall that the distance signless Laplacian matrix of G is defined to be $\mathcal{Q}(G) = Tr(G) + \mathcal{D}(G)$, where $\mathcal{D}(G)$ is the distance matrix, $Tr(G) = diag(D_1, D_2, \ldots, D_n)$ and D_i is the row sum of $\mathcal{D}(G)$ corresponding to vertex v_i. Denote by $\rho^{\mathcal{D}}(G)$, $\rho^{\mathcal{D}}_{min}(G)$ the largest eigenvalue and the least eigenvalue of $\mathcal{D}(G)$, respectively. And denote by $q^{\mathcal{D}}(G)$, $q^{\mathcal{D}}_{min}(G)$ the largest eigenvalue and the least eigenvalue of $\mathcal{Q}(G)$, respectively. The distance spread of a graph G is defined as $S_{\mathcal{D}}(G) = \rho^{\mathcal{D}}(G) - \rho^{\mathcal{D}}_{min}(G)$, and the distance signless Laplacian spread of a graph G is defined as $S_{\mathcal{Q}}(G) = q^{\mathcal{D}}(G) - q^{\mathcal{D}}_{min}(G)$. In this paper, we point out an error in the result of Theorem 2.4 in Yu et al. (2012) and modify it. As well, we obtain some lower bounds on distance signless Laplacian spread of a graph.

1. Introduction

Throughout this article, we assume that G is a simple, connected and undirected graph on n vertices. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. We denote by $deg(v_i)$ (simply, d_i) the degree of vertex v_i, and for $u, v \in V$, we denote by $d_G(u, v)$ the distance between u and v in G. Recall that the *distance matrix* is $\mathcal{D}(G) = (d_{ij})$ where $d_{ij} = d_G(v_i, v_j)$. For any $v_i \in V(G)$, the *transmission* of vertex v_i, denoted by $Tr_G(v_i)$ or D_i, is defined to be $\sum_{v_j \in V(G), j \neq i} d_G(v_i, v_j)$, which is equal to the row sum of $\mathcal{D}(G)$ corresponding to vertex v_i. Sometimes, D_i is called the *distance degree*. Let $Tr(G) = diag(D_1, D_2, \ldots, D_n)$ be the diagonal matrix of vertex transmissions of G. The *distance signless Laplacian matrix* of G is defined as $\mathcal{Q}(G) = Tr(G) + \mathcal{D}(G)$ (see [1]).

For a nonnegative real symmetric matrix M, we denote by $P_M(\lambda) = det(\lambda I - M)$ its the characteristic polynomial. Its largest eigenvalue is called the spectral radius of M. For a graph G, the *spectral radius* of $\mathcal{D}(G)$ and $\mathcal{Q}(G)$, denoted by $\rho^{\mathcal{D}}(G)$ and $q^{\mathcal{D}}(G)$, are also called the *distance spectral radius* and the *distance signless Laplacian spectral radius*, respectively. Denote by $\rho^{\mathcal{D}}_{min}(G)$ and $q^{\mathcal{D}}_{min}(G)$ the least eigenvalue of $\mathcal{D}(G)$ and the least eigenvalue of $\mathcal{Q}(G)$, respectively. The *distance spread* of graph G is defined as $S_{\mathcal{D}}(G) = \rho^{\mathcal{D}}(G) - \rho^{\mathcal{D}}_{min}(G)$, and the *distance signless Laplacian spread* of graph G is defined as $S_{\mathcal{Q}}(G) = q^{\mathcal{D}}(G) - q^{\mathcal{D}}_{min}(G)$. Without ambiguity, $S_{\mathcal{D}}(G)$ and $S_{\mathcal{Q}}(G)$ are shortened as $S_{\mathcal{D}}$ and $S_{\mathcal{Q}}$ sometimes.

From [9,12], we know that the spread of a matrix is a very interesting topic. As a result, in algebraic graph theory, the spread of some matrices of a graph also becomes interesting (see [6,11]). Because the research of the eigenvalues of the distance matrix of a graph is of great significance for both algebraic graph theory and practical applications, the problem

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concerning the distance spectrum of a graph has been studied extensively recently (see [2,3,5,7]). These cause the interests of the researchers on the problem about the distance spectral spread of a graph [10,14]. Motivated by these, in this paper, we proceed to consider the distance and distance signless Laplacian spreads of a graph.

In Section 3, we point out an error in the result of Theorem 2.4 in "Distance spectral spread of a graph" [G. Yu, etc., Discrete Applied Mathematics. 160 (2012) 2474–2478] and modify it. In Section 4, some lower bounds on distance signless Laplacian spread of a graph are shown.

2. Some preliminaries

In this section, we introduce some definitions, notations and working lemmas.

Let I_p be the $p \times p$ identity matrix and $J_{p,q}$ be the $p \times q$ matrix in which every entry is 1, or simply J_p if $p = q$. For a matrix M, its spectrum $\sigma(M)$ is the multiset of its eigenvalues.

Definition 2.1. Let M be a real matrix of order n described in the following block form

$$M = \begin{pmatrix} M_{11} & \cdots & M_{1t} \\ \vdots & \ddots & \vdots \\ M_{t1} & \cdots & M_{tt} \end{pmatrix}, \tag{2.1}$$

where the diagonal blocks M_{ii} are $n_i \times n_i$ matrices for any $i \in \{1, 2, \ldots, t\}$ and $n = n_1 + \cdots + n_t$. For any $i, j \in \{1, 2, \ldots, t\}$, let b_{ij} denote the average row sum of M_{ij}, i.e., b_{ij} is the result that the sum of all entries in M_{ij} is divided by the number of rows. Then $B(M) = (b_{ij})$ (simply by B) is called the quotient matrix of M.

Consider two sequences of real numbers: $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$ with $m < n$. The second sequence is said to interlace the first one whenever $\lambda_i \geq \mu_i \geq \lambda_{n-m+i}$ for $i = 1, 2, \ldots, m$.

Lemma 2.2 ([8]). *Let M be a symmetric matrix and have the block form as (2.1), B be the quotient matrix of M. Then the eigenvalues of B interlace the eigenvalues of M.*

Lemma 2.3 ([13]). *Let M be defined as (2.1), and for any $i, j \in \{1, 2 \ldots, t\}$, $M_{ii} = l_i J_{n_i} + p_i I_{n_i}$, $M_{ij} = s_{ij} J_{n_i,n_j}$ for $i \neq j$, where l_i, p_i, s_{ij} are real numbers, $B = B(M)$ be the quotient matrix of M. Then*

$$\sigma(M) = \sigma(B) \cup \{p_i^{[n_i-1]} \mid i = 1, 2 \ldots, t\}, \tag{2.2}$$

where $\lambda^{[t]}$ means that λ is an eigenvalue with multiplicity t.

By Lemma 2.3, we can obtain the distance (signless Laplacian) spectrum of $K_{a,b}$ as follows immediately, where $n = a + b$.

$$\sigma(\mathcal{D}(K_{a,b})) = \left\{(-2)^{[n-2]}, n - 2 \pm \sqrt{n^2 - 3ab}\right\}, \tag{2.3}$$

and

$$\sigma(\mathcal{Q}(K_{a,b})) = \left\{(2n - a - 4)^{[b-1]}, (2n - b - 4)^{[a-1]}, \frac{5n - 8 \pm \sqrt{9n^2 - 32ab}}{2}\right\}. \tag{2.4}$$

Lemma 2.4 ([4]). *Let H_n denote the set of all $n \times n$ Hermitian matrices, $A \in H_n$ with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, and B be a $m \times m$ principal matrix of A with eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$. Then $\lambda_i \geq \mu_i \geq \lambda_{n-m+i}$ for $i = 1, 2, \ldots, m$.*

3. Results on $S_{\mathcal{D}}$ for a bipartite graph

In [14], the authors obtained a lower bound for $S_{\mathcal{D}}(G)$ with the maximum degree Δ of G, but it is found that the result is incorrect when $\Delta \leq |V(G)| - 2$. In this section, we rectify it.

Let $G = (V, E)$ be a graph. For $v_i, v_j \in V$, if v_i is adjacent to v_j, we denote it by $v_i \sim v_j$ (simply, $i \sim j$). We let $t_v = \frac{\sum_{v_i \sim v} D_i}{d_v}$ be the *average distance degree* of v [14].

Proposition 3.1. *([14], Theorem 2.4) Let G be a simple connected bipartite graph on n vertices with $S = \sum_{i=1}^{n} D_i$ and maximum degree Δ. Suppose $deg(v_1) = deg(v_2) = \cdots = deg(v_k) = \Delta$. Then*

(i) *if $\Delta \leq n - 2$, we have*

$$S_{\mathcal{D}}(G) \geq \max_{1 \leq i \leq k} \frac{\sqrt{a_i^2 - 4b_i(\Delta + 1)(n - \Delta - 1)}}{(\Delta + 1)(n - \Delta - 1)}, \tag{3.1}$$

where $a_i = 2(n - t_{v_i} - 1)\Delta^2 + (S - 2t_{v_i} - 2)\Delta + S$ and $b_i = \Delta^2(2S - t_{v_i}^2 - 2t_{v_i} - 1)$.