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A sufficient condition for planar graphs with maximum degree 6 to be totally 8-colorable

Enqiang Zhu *, Jin Xu

School of Electronics Engineering and Computer Science, Peking University, Beijing, 100871, China
Key Laboratory of High Confidence Software Technologies (Peking University), Ministry of Education, China

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ABSTRACT

It is known that Total Coloring Conjecture (TCC) was verified for planar graphs with maximum degree $\Delta \neq 6$. In this paper, we prove that TCC holds for planar graphs G with $\Delta(G) = 6$, if G does not contain any subgraph isomorphic to a 4-fan.

1. Introduction

All graphs considered in this paper are simple, finite and undirected, and for the terminologies and notations not defined here we follow [2]. For any graph G, we denote by $V(G)$, $E(G)$, $\Delta(G)$ and $\delta(G)$ the *vertex set*, the *edge set*, the *maximum degree* and the *minimum degree* of G, respectively. For any vertex v in G, a vertex $u \in V(G)$ is said to be a neighbor of v if $uv \in E(G)$. We use $N_G(v)$ to denote the set of neighbors of v. The degree of v in G, denoted by $d_G(v)$, is the number of neighbors of v in G, i.e. $d_G(v) = |N_G(v)|$. A k-vertex or a k^+-vertex is a vertex of degree k or at least k. A k-neighbor of a vertex v is a neighbor of v with degree k. A *k-cycle* is a cycle of length k, and a 3-cycle is usually called a *triangle*. An (x, y, z)-triangle is a triangle whose vertices have degrees x, y and z.

A *partial total k-coloring f* of a graph G (*regarding to S*) is a coloring using k colors such that no two adjacent or incident elements in S get the same color, where $S \subseteq (V(G) \cup E(G))$. Particularly, when $S = V(G) \cup E(G)$, f is called a *total k-coloring* of G. A graph is *totally k-colorable* if it admits a total k-coloring. The *total chromatic number* of a graph G, denoted by $\chi_t(G)$, is the smallest integer k such that G has a total k-coloring.

It is clear that $\chi_t(G) \geq \Delta(G) + 1$. As for the upper bound, Behzad [1] and Vizing [9] proposed independently the famous Total Coloring Conjecture (TCC), claiming that $\chi_t(G) \leq \Delta(G) + 2$ for every simple graph G. So far TCC has been confirmed for graphs with $\Delta \leq 5$ [5], and for planar graphs with $\Delta \geq 7$ [3,4,7]. Therefore, the only open case for planar graphs is $\Delta = 6$. In [8], Sun et al. proved that every planar graph G with maximum degree 6 is totally 8-colorable if no two triangles in G share a common edge (which implies that every vertex v in G is incident with at most $\lfloor \frac{d_G(v)}{2} \rfloor$ triangles). In this paper, we give a stronger statement as follows.

Theorem 1.1. *Let G be a planar graph of maximum degree* 6. *If G does not contain any subgraph that is isomorphic to a* 4*-fan, as shown in Fig. 1, then G is totally* 8*-colorable.*

* Corresponding author at: School of Electronics Engineering and Computer Science, Peking University, Beijing, 100871, China.
E-mail address: zhuenqiang@pku.edu.cn (E. Zhu).

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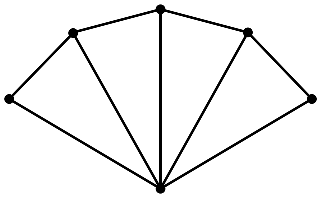

Fig. 1. A 4-fan.

The proof of this theorem generally runs as follows: We first explore the properties of a minimal counterexample (Section 2), by which we will obtain a contradiction via a discharging method (Section 3).

For convenience, let us first introduce the following notations: Let f be a partial total k-coloring of G regarding to $S, S \subseteq (V(G) \cup E(G))$. We use $\{1, 2, \ldots, k\}$ to denote the color set with k colors. The set of colors appearing on the edges incident with a vertex $v \in V(G)$ is denoted by $C_f(v)$, and $\overline{C}_f(v) = \{1, 2, \ldots, k\} \setminus C_f(v)$. Moreover, when $v \in S$, we denote by $C_f[v] = C_f(v) \cup \{f(v)\}$, and $\overline{C}_f[v] = \{1, 2, \ldots, k\} \setminus C_f[v]$. For any $X \subseteq S$, let $f(X) = \{f(x) : x \in X\}$. In what follows, we assume that a planar graph G is always embedded in the plane, and denote by $F(G)$ the set of faces of G. The degree of a face $f \in F(G)$, denoted by $d_G(f)$, is the number of edges incident with f, where each cut-edge is counted twice. A face of degree k is called a *k-face*. A face of degree at least k is a *k^+-face*.

2. Properties of a minimal counterexample

In this section, we investigate properties of a minimal counterexample, namely a planar graph with minimum sum of the number of edges and the number of vertices, which goes against Theorem 1.1. Let H be such a counterexample, then naturally H satisfies that:

(1) H is a planar graph of maximum degree 6.
(2) H does not contain any subgraph that is isomorphic to a 4-fan.
(3) H is not totally 8-colorable.
Moreover, since every planar graph with $\Delta \leq 5$ is totally 7-colorable [5] and every subgraph of H has property (2), we have the following property (4):
(4) Every proper subgraph of H is totally 8-colorable.

Apart from the above basic properties, we now deduce some structural properties of H.

Lemma 2.1. *H contains no edge uv with* $\min\{d_H(u), d_H(v)\} \leq 3$ *and* $d_H(u) + d_H(v) \leq 8$.

Proof. Suppose that H contains an edge uv with $d_H(u) \leq 3$ and $d_H(u) + d_H(v) \leq 8$. By the minimality of H, $H \setminus \{uv\}$ has a total 8-coloring f. Erase the color on u, and then color uv and u in turn. Since the number of colors that we cannot use is at most $2 + 5 = 7$ for uv and $3 + 3 = 6$ for u, it follows that f can be extended to a total 8-coloring of H. This contradicts the choice of H as a counterexample. □

According to Lemma 2.1, we can easily obtain the following result.

Corollary 2.2. *We have $\delta(H) \geq 3$, and for each $uv \in E(H)$, if $d_H(u) = 3$, then $d_H(v) = 6$.*

The proofs of the following results, Lemmas 2.3–2.5, are analogous to those in [6] (lemmas 7, 8 and 9), we include them here for the convenience of readers.

Lemma 2.3. *H has no triangle incident with a 3-vertex.*

Proof. Suppose that H contains a triangle $uvwu$ with $d_H(u) = 3$. Then, $d_H(v) = d_H(w) = 6$ by Lemma 2.1. Let f be a total 8-coloring of $H' = H \setminus \{uv\}$ after erasing the color of u. We will prove that f can be extended to a total 8-coloring of H, and a contradiction. Since u is a 3-vertex, it is sufficient to show that uv can be properly colored based on f. If $C_f(u) \cup C_f[v] \neq \{1, 2, \ldots, 8\}$, then uv can be colored with at least one available color in $\{1, 2, \ldots, 8\} \setminus (C_f(u) \cup C_f[v])$. If $C_f(u) \cup C_f[v] = \{1, 2, \ldots, 8\}$, then we without loss of generality assume that $C_f(u) = \{1, 2\}$, $C_f[v] = \{3, 4, 5, 6, 7, 8\}$, $f(uw) = 2$ and $f(vw) = 3$. Since w is a 6-vertex, it follows that $|\overline{C}_f[w]| = 1$. When $\overline{C}_f[w] = 1$, we can recolor vw with 1 and color uv with 3; When $\overline{C}_f[w] \neq 1$, we have $\overline{C}_f[w] \in \{4, 5, 6, 7, 8\}$. Then, we can recolor uw with $\overline{C}_f[w]$ and color uv with 2. □

Lemma 2.4. *H does not contain any triangle incident with two 4-vertices.*