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Trees with equal total domination and game total domination numbers

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ABSTRACT

In this paper, we continue the study of the total domination game in graphs introduced in Henning et al. (2015), where the players Dominator and Staller alternately select vertices of G. Each vertex chosen must strictly increase the number of vertices totally dominated, where a vertex totally dominates another vertex if they are neighbors. This process eventually produces a total dominating set S of G in which every vertex is totally dominated by a vertex in S. Dominator wishes to minimize the number of vertices chosen, while Staller wishes to maximize it. The game total domination number, $\gamma_{\rm tg}(G)$, (respectively, Staller-start game total domination number, $\gamma'_{\rm tg}(G)$) of G is the number of vertices chosen when Dominator (respectively, Staller) starts the game and both players play optimally. For general graphs G, sometimes $\gamma_{\rm tg}(G) > \gamma'_{\rm tg}(G)$. We show that if G is a forest with no isolated vertex, then $\gamma_{\rm tg}(G) \le \gamma'_{\rm tg}(G)$. Using this result, we characterize the trees with equal total domination and game total domination number.

1. Introduction

The domination game in graphs was first introduced by Brešar, Klavžar, and Rall [4] and extensively studied afterwards in [1,3,5–7,10,11,13,14,17,20–22] and elsewhere. Recently, the total version of the domination game was investigated in [15], where it was demonstrated that these two versions differ significantly. The total version has been studied in [2,8,9, 12,16,18] and elsewhere. A vertex *totally dominates* another vertex if they are neighbors. A *total dominating set*, abbreviated TD-set, of a graph G is a set S of vertices such that every vertex of G is totally dominated by a vertex in S. The *total domination game* consists of two players called *Dominator* and *Staller*, who take turns choosing a vertex from G. Each vertex chosen must totally dominate at least one vertex not totally dominated by the set of vertices previously chosen. Following the notation of [15], we call such a chosen vertex a *legal move* or a *playable vertex* in the total domination game. The game ends when the set of vertices chosen is a total dominating set in G. Thus we will assume that all graphs under consideration in this paper have minimum degree at least 1. Dominator's objective is to minimize the number of vertices chosen, while Staller's is to end the game with as many vertices chosen as possible.

The *Dominator-start total game* is the total domination game when Dominator starts the game, while the *Staller-start total game* is the total domination game when Staller starts the game. The *game total domination number*, $\gamma_{\rm tg}(G)$, of G is the number of vertices chosen in the Dominator-start total game when both players employ a strategy that achieves their objective. The number of vertices chosen in the Staller-start total game when both players employ a strategy that achieves

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their objective is the *Staller-start game total domination number*, $\gamma'_{\rm tg}(G)$, of G. Determining the exact value of $\gamma_{\rm tg}(G)$ and $\gamma'_{\rm tg}(G)$ is a challenging problem, and is currently known only for paths and cycles [12].

Since the exact values of these invariants are very difficult to compute, we will often employ the so-called *imagination strategy* when it is required to show that the game total domination number of a tree and one of its subtrees differs by exactly (or by at most) some fixed amount. This method of proof was introduced in the initial paper [4] on game domination. Here it consists of both Dominator and Staller playing the total domination game on one of the trees while Dominator "imagines" the game being played on the second tree. Dominator chooses legal moves in the second tree that are in keeping with his objective of minimizing the total number of vertices chosen there. His moves in this second tree are then either copied directly to the original tree (where the "real" game is being played by both players) or modified in some way so as to be legal moves in the real game. See [4] for further explanation of this proof technique.

A *partially total dominated graph* is a graph together with a declaration that some vertices are already totally dominated; that is, they need not be totally dominated in the rest of the game. Given a graph G and a subset S of vertices of G, we denote by $G|S$ the partially total dominated graph in which the vertices of S in G are already totally dominated. We use $\gamma_{\rm tg}(G|S)$ (resp. $\gamma'_{\rm tg}(G|S)$) to denote the number of turns remaining in the game on $G|S$ under optimal play when Dominator (resp. Staller) has the next turn. In [15], the authors present a key lemma, named the *Total Continuation Principle*.

Lemma 1 (*"Total Continuation Principle" — [15, Lemma 2.1]*). *Let G be a graph and let $A, B \subseteq V(G)$. If $B \subseteq A$, then $\gamma_{\rm tg}(G|A) \le \gamma_{\rm tg}(G|B)$ and $\gamma'_{\rm tg}(G|A) \le \gamma'_{\rm tg}(G|B)$.*

As a consequence of the Total Continuation Principle, when the total domination game is played on a partially total dominated graph G, the numbers $\gamma_{\rm tg}(G)$ and $\gamma'_{\rm tg}(G)$ can differ by at most 1.

Corollary 1 (*[15]*). *For every graph G with no isolated vertex, we have $|\gamma_{\rm tg}(G) - \gamma'_{\rm tg}(G)| \le 1$.*

1.1. Notation

For notation and graph theory terminology not defined herein, we in general follow [19]. We denote the *degree* of a vertex v in a graph G by $d_G(v)$, or simply by $d(v)$ if the graph G is clear from the context. A *degree-k vertex* is a vertex of degree k. The minimum degree among the vertices of G is denoted by $\delta(G)$. A vertex of degree 1 is called a *leaf* and its neighbor a *support vertex*. A *strong support vertex* is a support vertex with at least two leaf neighbors. A star is a tree with at most one vertex of degree 2 or more. A *subdivided star* is the tree obtained from a star on at least three vertices by subdividing every edge exactly once. The *open neighborhood* of a vertex $v \in V(G)$ is $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$ and the degree of v is $d_G(v) = |N_G(v)|$. The *closed neighborhood of* v is $N_G[v] = \{v\} \cup N_G(v)$.

For a set $S \subseteq V(G)$, we let $G[S]$ denote the subgraph induced by S. The graph obtained from G by deleting the vertices in S and all edges incident with vertices in S is denoted by $G - S$. If $S = \{v\}$, we also denote $G - S$ simply by $G - v$.

If X and Y are subsets of vertices in a graph G, then the set X *totally dominates* the set Y in G if every vertex of Y is adjacent to at least one vertex of X. In particular, if X totally dominates the vertex set of G, then X is a TD-set in G. The cardinality of a smallest TD-set in G is the *total domination number* of G and is denoted $\gamma_t(G)$. A TD-set of G of cardinality $\gamma_t(G)$ is called a *$\gamma_t(G)$-set*. Since an isolated vertex in a graph cannot be totally dominated by definition, all graphs considered will be without isolated vertices. For more information on total domination in graphs see the recent book [19]. We use the standard notation $[k] = \{1, \dots, k\}$.

A *rooted tree* T distinguishes one vertex r called the *root*. For each vertex $v \neq r$ of T, the *parent* of v is the neighbor of v on the unique (r, v)-path, while a *child* of v is any other neighbor of v. We denote all the children of a vertex v by $C(v)$. A *descendant* of v is a vertex $u \neq v$ such that the unique (r, u)-path contains v. Thus, every child of v is a descendant of v. An *ancestor* of v is a vertex $u \neq v$ that belongs to the (r, v)-path in T. In particular, the parent of v is an ancestor of v. The *grandparent* of v is the ancestor of v at distance 2 from v. A *grandchild* of v is a descendant of v at distance 2 from v. A path on n vertices is denoted by P_n.

Let G be a partially total dominated graph and let v be a vertex of G. If v is not totally dominated in G, we call the vertex v *totally undominated* in G. We let G_v denote the partially total dominated graph obtained from G by totally dominating $N(v)$. If the vertex v is totally dominated in G, then we let G^v denote the partially total dominated graph obtained from G by removing v from the set of totally dominated vertices. We note that G^v and G are identical except that v is totally dominated in G but not in G^v.

2. Main result

As remarked by Brešar, Klavžar, Košmrlj, and Rall [3], "the domination game is very non-trivial even when played on trees". In this paper we prove the following result.

Theorem 1. *If F is a partially total dominated forest with no isolated vertex, then $\gamma_{\rm tg}(F) \le \gamma'_{\rm tg}(F)$.*

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