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Dense on-line arbitrarily partitionable graphs☆

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ABSTRACT

A graph G of order n is called *arbitrarily partitionable* (AP, for short) if, for every sequence $(n_1, \dots, n_k)$ of positive integers with $n_1+\dots+n_k = n$, there exists a partition $(V_1, \dots, V_k)$ of the vertex set $V(G)$ such that V_i induces a connected subgraph of order n_i, for $i = 1, \dots, k$. In this paper we consider the on-line version of this notion, defined in a natural way.

We prove that if G is a connected graph such with the independence number at most $\lceil \frac{n}{2} \rceil$ and the degree sum of any pair of non-adjacent vertices is at least $n-3$, then G is on-line arbitrarily partitionable except for two graphs of small orders. We also prove that if G is a connected graph of order n and size $\|G\| > \binom{n-3}{2} + 6$, then G is on-line AP unless n is even and G is a spanning subgraph of a unique exceptional graph. These two results imply that dense AP graphs satisfying one of the above two assumptions are also on-line AP. This is in contrast to sparse graphs where only few AP graphs are also on-line AP.

1. Introduction

We use standard graph theory terminology and notation. The number of edges of a graph G is called the *size* of G and is denoted by $\|G\|$. A graph G is called *traceable* if it contains a *Hamiltonian path*, i.e. a path through all vertices of G. By $c(G)$ we denote the *circumference* of a graph G, i.e. the length of a longest cycle. If C is a cycle with a given orientation and $x \in V(C)$, then by x^+ and x^- we denote a successor and a predecessor of x along the orientation of C. We also use the notation

$$\sigma_2(G) = \min\{d(x) + d(y) \; : \; xy \notin E(G)\}.$$

If G_1 and G_2 are two graphs with disjoint vertex sets, then by $G_1 \vee G_2$ we denote their *join*, that is a graph with the vertex set $V(G_1) \cup V(G_2)$ and the edge set $E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup \{uv \; : \; u \in V(G_1), v \in V(G_2)\}$.

Let us now introduce some terminology for the problem we deal with. If $G = (V, E)$ is a graph of order n, then a sequence $\tau = (n_1, \dots, n_k)$ of positive integers is called *admissible for* G if $n_1 + \cdots + n_k = n$. Such an admissible sequence τ is said to be *realizable in* G if the vertex set V can be partitioned into k parts $(V_1, \dots, V_k)$ such that $|V_i| = n_i$ and the subgraph $G[V_i]$ induced by V_i is connected, for every $i = 1, \dots, k$. We say that G is *arbitrarily partitionable* (AP, for short) if every admissible sequence τ for G is realizable in G.

The notion of AP graphs was introduced by Barth, Baudon and Puech in [1] (and independently by Horňák and Woźniak in [11]), motivated by a problem concerning graphs modelling parallel systems (parallel computing, networks of workstations, etc.), considered as networks connecting different computing resources. Suppose there are k users, where the ith one needs n_i resources from our network. The subgraph induced by the set of resources attributed to each user

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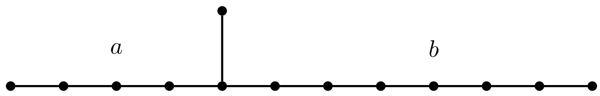

Fig. 1. Cat(a, b) with $a = 5$ and $b = 8$.

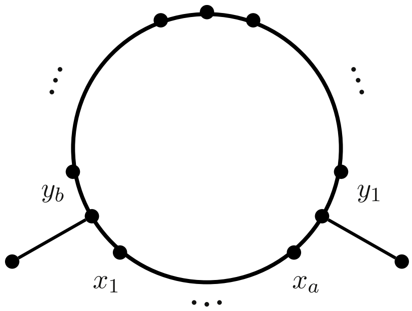

Fig. 2. Sun (a, b) with two rays.

Table 1
Values a, b such that Cat(a, b) is on-line AP.

a	b
2	$\equiv 1 \pmod 2$
3	$\equiv 1, 2 \pmod 3$
4	$\equiv 1 \pmod 2$
5	6, 7, 9, 11, 14, 19
6	$\equiv 1, 5 \pmod 6$
7	8, 9, 11, 13, 15
8	11, 19
9	11
10	11
11	12

should be connected and each resource should be attributed to one user. So we are seeking a realization of the sequence $\tau = (n_1, \ldots, n_k)$ in this graph. Suppose that we want to do it for any number of users and any sequence of request. Thus, such a network should be an AP graph.

The general concept of arbitrarily partitionable graphs, sometimes also called *arbitrarily vertex decomposable* [11] or *fully decomposable* [7] or just *decomposable* [1], has spawned some thirty papers and is still developing (examples of recent papers are [3,4,7,12,19]). Here we quote only the results we directly use in this paper.

By Cat(a, b), where $2 \le a \le b$, we denote a caterpillar with three leaves obtained from the star $K_{1,3}$ by substituting two of its edges by paths of orders a and b, respectively. Fig. 1 shows Cat (5, 8). One of the earliest results about AP graphs is the following one proved independently by the authors of this concept.

Theorem 1 ([1,11]). *The caterpillar* Cat (a, b) *is* AP *if and only if* a *and* b *are relatively prime.*

A *sun with r rays* is a graph of order $n \ge 2r$ with r independent hanging edges, called *rays*, whose deletion yields a cycle C_{n-r}. A sun with two rays (cf. Fig. 2) such that the deletion of vertices of degree three divides C_{n-r} into two paths of orders a and b, where $0 \le a \le b$, is denoted by Sun (a, b).

In [13], we found all AP suns with at most three rays. Our result for suns with two rays follows.

Theorem 2 ([13]). Sun (a, b) *is* AP *if and only if at least one of the numbers* a, b *is even.*

The definition of AP graphs has many variations. One of them, even more natural from the point of view of applications to computer science, is the following concept of on-line arbitrarily partitionable graphs introduced by Horňák, Tuza and Woźniak in [10]. The definition is natural. We are not given a whole admissible sequence at the beginning but we get its elements one by one, and each time we have to choose a connected subgraph of prescribed order, having no possibility to change this choice later. If this procedure can be accomplished for any admissible sequence, then the graph is called *on-line arbitrarily partitionable* (on-line AP, for short).

Horňák, Tuza and Woźniak [10] characterized all on-line AP trees.

Theorem 3 ([10]). *A tree T is on-line* AP *if and only if T is either a path or a caterpillar* Cat (a, b) *with a and b given in Table 1 or an exceptional tree that results from gluing together three paths of lengths* 3, 5, 7 *at one of their endpoints.*