ARTICLE IN PRESS

Discrete Applied Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

On partitions of graphs under degree constraints

Muhuo Liu [a,b], Baogang Xu [c,*]

[a] Department of Mathematics, College of Mathematics and Informatics, South China Agricultural University, Guangzhou, 510642, China
[b] College of Mathematics and Statistics, Shenzhen University, Shenzhen, 518060, China
[c] Institute of Mathematics, School of Mathematical Sciences, Nanjing Normal University, Nanjing, 210023, China

ARTICLE INFO

Article history:
Received 4 October 2014
Received in revised form 20 March 2017
Accepted 6 April 2017
Available online xxxx

Keywords:
Partition
$(K_4 - e)$-free graphs
Subgraph
Degree

ABSTRACT

Let s, t be two integers, and let $g(s, t)$ denote the minimum integer such that the vertex set of a graph of minimum degree at least $g(s, t)$ can be partitioned into two nonempty sets which induce subgraphs of minimum degree at least s and t, respectively. In this paper, it is shown that, (1) for positive integers s and t, $g(s, t) \leq s+t$ on $(K_4 - e)$-free graphs except K_3, and (2) for integers $s \geq 2$ and $t \geq 2$, $g(s, t) \leq s+t-1$ on triangle-free graphs in which no two quadrilaterals share edges. Our first conclusion generalizes a result of Kaneko (1998), and the second generalizes a result of Diwan (2000).

1. Introduction

All graphs considered in this paper are finite and simple. Let G be a graph, let X be a subset of $V(G)$. We use $G[X]$ to denote the subgraph of G induced by X. For a vertex $x \in X$, we use $N_X(x)$ to denote the neighbor set of x in X, let $N_X[x] = N_X(x) \cup \{x\}$, and let $d_X(x) = |N_X(x)|$ (when $X = V(G)$, we simplify $N_X(x)$, $N_X[x]$ and $d_X(x)$ as $N(x)$, $N[x]$ and $d(x)$, respectively). Let A and B be two nonempty disjoint subsets of $V(G)$. If $A \cup B = V(G)$, then we call (A, B) a partition of $V(G)$. We also say that $V(G)$ is partitioned into A and B if (A, B) is a partition.

In 1972, Mader [6] showed that every graph of minimum degree at least $4k$ contains a k-connected subgraph. In 1982, Györi (see [4,8]) proposed a problem as follows: for given positive integers s and t, is there an integer $f(s, t)$ such that the vertex set of every $f(s, t)$-connected graph can be partitioned into two sets S and T which induce subgraphs of connectivity at least s and t respectively? Thomassen [8], and Szegedy independently (see [4]), proved the existence of the function $f(s, t)$, and Hajnal [4] improved the bound to $f(s, t) \leq 4s + 4t - 13$. In his proof, Thomassen proved a degree version of Györi's problem. He showed essentially that for positive integers s and t, there is an integer $g(s, t)$ such that the vertex set of every graph G with minimum degree at least $g(s, t)$ can be partitioned into S and T which induce subgraphs of minimum degree at least s and t, respectively. The complete graph K_{s+t+1} shows that $g(s, t) \geq s + t + 1$. Then, Thomassen conjectured that $g(s, t) = s + t + 1$.

In [7], Stiebitz confirmed Thomassen's conjecture with an elegant argument. In fact, Stiebitz proved a result stronger than the conjecture. Let $\mathbb{N}$ denote the set of nonnegative integers.

Theorem 1.1 *([7]). Let G be a graph and $a, b : V(G) \longmapsto \mathbb{N}$ two functions. Suppose that $d(x) \geq a(x) + b(x) + 1$ for each vertex x of G. Then, there exists a partition of $V(G)$ into A and B such that*

(1) *$d_A(x) \geq a(x)$ for each $x \in A$, and*
(2) *$d_B(y) \geq b(y)$ for each $y \in B$.*

* Corresponding author.
E-mail addresses: liumuhuo@163.com (M. Liu), baogxu@njnu.edu.cn (B. Xu).

http://dx.doi.org/10.1016/j.dam.2017.04.007
0166-218X/

Please cite this article in press as: M. Liu, B. Xu, On partitions of graphs under degree constraints, Discrete Applied Mathematics (2017), http://dx.doi.org/10.1016/j.dam.2017.04.007.

Let (A, B) be a partition of $V(G)$, and let $a, b : V(G) \longmapsto \mathbb{N}$ be two functions. We say that (A, B) is an (a, b)-*feasible partition* if $d_A(x) \geq a(x)$ for each $x \in A$ and $d_B(y) \geq b(y)$ for each $y \in B$. Theorem 1.1 says that G admits an (a, b)-feasible partition if $d(x) \geq a(x) + b(x) + 1$ for each vertex x of G. Stiebitz [7] further asked a question if there are some pair of positive integers s and t and a triangle-free graph G of minimum degree $s + t$ such that G has no vertex disjoint subgraphs G_1 and G_2 with minimum degree at least s and t, respectively. In another words, is it true that, for any positive integers s and t, $g(s, t) \leq s+t$ on triangle-free graphs? The complete bipartite graph $K_{s+t,s+t}$ shows that $g(s, t) \geq s + t$ on triangle-free graphs, and every connected regular triangle-free graph requires s and t to be positive in order to have $g(s, t) \leq s + t$. Kaneko [5] answered Stiebitz's problem with a similar argument as that used in [7].

Theorem 1.2 (*[5]*). *Let s and t be two positive integers. Then,* $g(s, t) \leq s + t$ *on triangle-free graphs.*

As Stiebitz pointed out in his paper [7], K_{s+t+1} does not admit (s, t)-feasible partitions for any pair $s \geq 1$ and $t \geq 1$, and the icosahedron is 5-regular and does not admit $(4, 1)$-feasible partitions. Note that triangles appear densely in both K_{s+t+1} and the icosahedron (every set of three vertices of K_{s+t+1} spans a triangle, and every edge of the icosahedron is in two triangles). One may ask naturally whether the bound $g(s, t) \leq s+t$ holds on graphs in which the triangles are not dense. This is indeed the case. A cycle of length 4 is referred to as a *quadrilateral*, and $K_4 - e$ is the graph obtained from K_4 by removing one edge. A graph is said to be $(K_4 - e)$-*free* if it does not contain $K_4 - e$ as a subgraph (here $K_4 - e$ may not be induced, a $(K_4 - e)$-free graph is also K_4-free. The similar happens when we talk no two quadrilaterals sharing an edge later). We show that $g(s, t) \leq s + t$ on $(K_4 - e)$-free graphs except K_3.

Theorem 1.3. *Let G be a* $(K_4 - e)$*-free graph with at least four vertices, and* $a, b : V(G) \longmapsto \mathbb{N} \setminus \{0\}$ *two functions. If* $d(x) \geq a(x) + b(x)$ *for each vertex x of G, then G admits an* (a, b)*-feasible partition.*

The requirement $(K_4 - e)$-free is necessary in Theorem 1.3 as evidenced by the icosahedron. Another example is $K_4 - e$ itself. Let $G = K_4 - e$, and let $a, b : V(G) \longmapsto \mathbb{N} \setminus \{0\}$ be two functions such that $a(x) = d(x) - 1$ and $b(x) = 1$ for each vertex $x \in V(G)$. Then G has no (a, b)-feasible partition.

As usual, the length of a shortest cycle in a graph G is called the *girth* of G. In 2000, Diwan considered the problem that whether $g(s, t)$ can be reduced further by forbidding the existence of triangles and quadrilaterals in the graphs, and he succeeded in showing that

Theorem 1.4 (*[2]*). *Let* $s \geq 2$ *and* $t \geq 2$ *be two integers. Then,* $g(s, t) \leq s + t - 1$ *on the graphs of girth at least five.*

The cycle of length n $(n \geq 5)$ shows that one cannot expect to omit the requirement of $s \geq 2$ and $t \geq 2$ by simply increasing the girth of graphs. In 2004, Gerber and Kobler generalized Theorem 1.4 and proved the following analogue of Theorem 1.1. Bazgan, Tuza and Vanderpooten [1] presented three polynomial time algorithms to find (a, b)-feasible partitions satisfying Theorems 1.1, 1.2 and 1.5, respectively.

Theorem 1.5 (*[3]*). *Let G be a graph of girth at least five, and* $a, b : V(G) \longmapsto \mathbb{N} \setminus \{0, 1\}$ *two functions. If* $d(x) \geq a(x) + b(x) - 1$ *for each vertex x of G, then G admits an* (a, b)*-feasible partition.*

Our next result generalizes Theorem 1.5 to triangle-free graphs that may contain quadrilaterals.

Theorem 1.6. *Let G be a triangle-free graph in which no two quadrilaterals share edges, and* $a, b : V(G) \longmapsto \mathbb{N} \setminus \{0, 1\}$ *two functions. If* $d(x) \geq a(x) + b(x) - 1$ *for each vertex x of G, then G admits an* (a, b)*-feasible partition.*

The complete bipartite graph $K_{3,3}$ shows that the restriction on the sparsity of quadrilaterals cannot be relaxed too much, since it does not admit $(2, 2)$-feasible partitions. We are not sure whether Theorem 1.6 can be improved further. It would be nice if someone can strengthen Theorem 1.6 to graphs with neither triangle nor $K_{2,3}$. Furthermore, up to our best knowledge, the following problem due to Diwan [2] is still open: whether the bound $s + t - 1$ in Theorem 1.4 can be improved further for graphs with larger girth.

As a direct corollary of Theorems 1.3 and 1.6, we have

Corollary 1.1. *Let s and t be two positive integers. Then,* $g(s, t) \leq s + t$ *on* $(K_4 - e)$*-free graphs except* K_3*, and* $g(s, t) \leq s + t - 1$ *on triangle-free graphs in which no two quadrilaterals share edges if* $s \geq 2$ *and* $t \geq 2$*.*

Before proving our theorems, we still need to introduce some notations that are also used in [1–3,5,7]. Let G be a graph, and let S be a subset of $V(G)$. Recall that for each vertex x of S, $d_S(x)$ denotes the degree of x in $G[S]$. Let y be a vertex in $V(G) \setminus S$. We use $e_G(y, S)$ to denote the number of edges joining y to S.

Let $a, b : V(G) \longmapsto \mathbb{N}$ be two functions. We say that S is a-*satisfactory* if $d_S(x) \geq a(x)$ for each vertex x of S, and say that S is a-*degenerate* if for each nonempty subset S' of S there exists a vertex $x \in S'$ such that $d_{S'}(x) \leq a(x)$. By an (a, b)-*degenerate partition* we mean a partition (A, B) of $V(G)$ such that A is a-degenerate and B is b-degenerate.

As in [3,7,8], the weight $\omega(A, B)$ of an (a, b)-degenerate partition (A, B) is defined by

$$\omega(A, B) = |E(G[A])| + |E(G[B])| + \sum_{u \in A} b(u) + \sum_{v \in B} a(v).$$