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On edge intersection graphs of paths with 2 bends☆

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ABSTRACT

An EPG-representation of a graph G is a collection of paths in the plane square grid, each corresponding to a single vertex of G, so that two vertices are adjacent if and only if their corresponding paths share infinitely many points. In this paper we focus on graphs admitting EPG-representations by paths with at most 2 bends. We show hardness of the recognition problem for this class of graphs, along with some subclasses.

We also initiate the study of graphs representable by unaligned polylines, and by polylines, whose every segment is parallel to one of prescribed slopes. We show hardness of recognition and explore the trade-off between the number of bends and the number of slopes.

1. Introduction

The concept of *edge intersection graphs of paths in a grid* (*EPG-graphs*) was introduced by Golumbic et al. [12]. By an EPG-representation of a graph G we mean a mapping from vertices of G to paths in the plane square grid, such that two vertices are adjacent if and only if their corresponding paths share a grid edge. As each graph can be represented in this way (see Golumbic et al. [12]), it makes sense to consider representations with some restricted sets of shapes. A usual parameterization is by bounding the number k of times each path is allowed to change the direction. Graphs with such a representation are called *k-bend graphs*. There are two main branches in this kind of research. The first one is understanding the structure of graphs with at most k bends — so far, the case of 1-bend graphs received most attention [12,6,10,1]. The other is finding the smallest k, such that every graph of a given class $\mathcal{G}$ is a k-bend graph. The most interesting results seem to concern planar graphs [4,14].

Since 0-bend graphs are just *interval graphs*, they can be recognized in polynomial time (see e.g. Booth and Lueker [5]). The recognition of 1-bend graphs is NP-complete (see Heldt et al. [13]), even if the representation is restricted to any prescribed set of 1-bend objects (see Cameron et al. [6]). However, the problem becomes trivially solvable when k is at least the maximum degree of the input graph [13]. Thus it is unclear whether k-bend graphs are hard to recognize for all $k \geq 2$.

It is worth mentioning the closely related notion of *B_k-VPG-graphs*. These graphs are defined as intersection graphs of axis-aligned paths with at most k bends. So, unlike in the EPG-representation, paths that share a finite number of points define adjacent vertices. Chaplick et al. [8] showed it is NP-complete to recognize B_k-VPG-graphs, for all $k \geq 0$.

☆ The extended abstract of this paper was presented on the conference WG 2016 [16].

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In this paper we explore the problem of recognition of subclasses of EPG-graphs. Namely, we show that it is NP-complete to recognize 2-bend graphs. We also consider some restrictions, where we permit just some types of the curves in an EPG-representation (similarly to [6]). One of these restrictions, i.e., *monotonic EPG-representations*, where each path ascends in rows and columns, was already considered by Golumbic et al. [12]. Our hardness proof even shows that between monotonic 2-bend graphs and 2-bend graphs, no polynomially recognizable class can be found.

The class of 2-bend graphs can be perceived as a generalization of the quite well-studied class of 1-bend graphs. We also consider some generalizations of the concept of EPG-representations. We do not require individual segments to be axis-aligned, but we permit them to use any slope. We call such graphs *edge-intersection graphs of polylines (EP)* and study the number of bends in this setting. After this generalization, we may ask about particular restrictions. These restrictions are represented by restricting number of slopes that segments may use or even by using just prescribed shapes (in a flavor similar to [6]).

For EP-graphs, we show that it is NP-hard to determine whether a graph is an unaligned 2-bend graph (hardness of the recognition for 1-bend graphs follows from [6]).

Having introduced EP-graphs, we observe that there is some trade-off between the number of bends and the number of slopes used in a representation. We also show that given an unaligned 2-bend graph on n vertices, we may need $\Omega(\sqrt{n})$ slopes to represent it. This result appears to be a corollary of our hardness-reduction.

The paper is organized as follows. We start with some definitions and preliminary observations on the structure of 2-bend graphs. In Section 3 we prove that it is NP-complete to recognize the graphs from this class and also from some of its subclasses. In Section 4 we introduce EP-graphs and show NP-hardness of recognition of unaligned 2-bend graphs. Then we show a lower bound on the number of slopes required for the representation of any unaligned 2-bend graph on n vertices and we discuss the relations between the number of bends and the number of slopes. The paper is concluded with some open questions in Section 5.

The extended abstract of this paper was presented on the conference WG 2016 [16].

2. Preliminaries

For an EPG-representation of a graph G, by P_v we shall denote the path representing a vertex v. Often we shall identify the vertex v with the path P_v. For example, if we say that two paths are *adjacent*, we mean that they share infinitely many points. Note that if two paths *intersect*, one common point is enough.

A central notion in the study of EPG-graphs is the *bend number*. The bend number of a graph G, denoted by $b(G)$, is the minimum k, such that G has an EPG-representation, in which every path changes it direction at most k times. Without loss of generality we can assume that every path in a k-bend EPG-representation bends exactly k times.

Each 2-bend path will be classified as *vertical* or *horizontal*, if its middle segment is respectively vertical or horizontal. This middle segment will be called the *body* of the path, while the remaining two segments will be referenced as its *legs*.

For a set X of shapes of polylines (i.e., piecewise-linear curves), by X-graphs we shall denote the class of graphs admitting an EPG-representation, in which the shape of every path is in X (similar notation was used in [6]). So for example 1-bend graphs are {⌞, ⌟, ⌜, ⌝}-graphs, while monotonic 2-bend graphs are exactly {_⌐, ⌐'}-graphs.

Golumbic et al. [12] analyzed the structure of cliques in 1-bend graphs and proved that in 1-bend graphs each clique C is either an *edge-clique* or a *claw-clique*. A maximal edge-clique consists of vertices whose representing paths share a common grid edge. A *claw* is a set of three distinct grid edges sharing a single endpoint and a maximal claw-clique consists of all paths containing two out of three edges of a given claw. Since we can safely assume that each 1-bend representation of a graph with n vertices can be embedded in a $2n \times 2n$ grid (see also [12,2]), we obtain that the number of maximal cliques in a 1-bend graph is at most $O(n^2)$, i.e., is polynomial in n. This is no longer the case with 2-bend graphs.

Let n be a positive integer and let K_{2n}^- be the *cocktail-party graph*, i.e., a complete graph on $2n$ vertices with a perfect matching removed. It is clear that K_{2n}^- has $2^n = 2^{|V(K_{2n}^-)|/2}$ maximal cliques. Fig. 1 (left) shows that K_{2n}^- is a 2-bend graph. Thus we obtain the following.

Proposition 1. 2-*bend graphs can have exponentially many maximal cliques.*

The restricted structure of cliques in 1-bend graphs follows from the fact that the 1-bend paths representing pairwise adjacent vertices must all share at least one grid point. It is easy to observe that cliques in 2-bend graphs do not have such a simple structure. One could be inclined by Fig. 1 (left) that every maximal clique is contained in the union of two edge-cliques or claw-cliques (a similar situation appears in unit disk graphs and is the main ingredient of a polynomial algorithm for Clique in these graphs — see Clark et al. [9]). However, Fig. 1 (right) shows it is not true.

3. Aligned 2-bend graphs

The main result of this section is the following complexity result.

Theorem 2. *It is* NP-*complete to decide if a given graph is a* 2-*bend graph.*

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