Discrete Applied Mathematics 226 (2017) 117–126

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Further results on the deficiency of graphs

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ARTICLE INFO

Article history:
Received 2 August 2016
Received in revised form 12 March 2017
Accepted 6 April 2017
Available online 10 May 2017

Dedicated to the memory of Haroutiun Khachatrian

Keywords:
Proper edge-coloring
Interval (consecutive) coloring
Deficiency
Complete graph
Near-complete graph

ABSTRACT

A *proper t-edge-coloring* of a graph G is a mapping $\alpha : E(G) \to \{1, \ldots, t\}$ such that all colors are used, and $\alpha(e) \neq \alpha(e')$ for every pair of adjacent edges $e, e' \in E(G)$. If α is a proper edge-coloring of a graph G and $v \in V(G)$, then the *spectrum of a vertex* v, denoted by $S(v, \alpha)$, is the set of all colors appearing on edges incident to v. The *deficiency of* α *at vertex* $v \in V(G)$, denoted by $def(v, \alpha)$, is the minimum number of integers which must be added to $S(v, \alpha)$ to form an interval, and the *deficiency* $def(G, \alpha)$ *of a proper edge-coloring* α *of* G is defined as the sum $\sum_{v\in V(G)} def(v, \alpha)$. The *deficiency of a graph* G, denoted by $def(G)$, is defined as follows: $def(G) = \min_\alpha def(G, \alpha)$, where minimum is taken over all possible proper edge-colorings of G. For a graph G, the smallest and the largest values of t for which it has a proper t-edge-coloring α with deficiency $def(G, \alpha) = def(G)$ are denoted by $w_{def}(G)$ and $W_{def}(G)$, respectively. In this paper, we obtain some bounds on $w_{def}(G)$ and $W_{def}(G)$. In particular, we show that for any $l \in \mathbb{N}$, there exists a graph G such that $def(G) > 0$ and $W_{def}(G) - w_{def}(G) \geq l$. It is known that for the complete graph K_{2n+1}, $def(K_{2n+1}) = n$ $(n \in \mathbb{N})$. Recently, Borowiecka-Olszewska, Drgas-Burchardt and Hałuszczak posed the following conjecture on the deficiency of near-complete graphs: if $n \in \mathbb{N}$, then $def(K_{2n+1} - e) = n - 1$. In this paper, we confirm this conjecture.

1. Introduction

All graphs considered in this paper are finite, undirected, and have no loops or multiple edges. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of G, respectively. The degree of a vertex $v \in V(G)$ is denoted by $d_G(v)$, the diameter of G by $\mathrm{diam}(G)$, and the chromatic index of G by $\chi'(G)$. For a graph G, let $\Delta(G)$ and $\delta(G)$ denote the maximum and minimum degrees of vertices in G, respectively. The terms and concepts that we do not define can be found in [2,24,37].

A *proper edge-coloring* of a graph G is a mapping $\alpha : E(G) \to \mathbb{N}$ such that $\alpha(e) \neq \alpha(e')$ for every pair of adjacent edges $e, e' \in E(G)$. If the set of colors is $\{1, \ldots, t\}$ and all colors are used, then α is called a *proper t-edge-coloring* of G. If α is a proper edge-coloring of a graph G and $v \in V(G)$, then the *spectrum of a vertex* v, denoted by $S(v, \alpha)$, is the set of all colors appearing on edges incident to v. A proper t-edge-coloring α of a graph G is an *interval t-coloring* if for each vertex v of G, the set $S(v, \alpha)$ is an interval of integers. A graph G is *interval colorable* if it has an interval t-coloring for some positive integer t. The set of all interval colorable graphs is denoted by $\mathfrak{N}$. For a graph $G \in \mathfrak{N}$, the smallest and the largest values of t for which it has an interval t-coloring are denoted by $w(G)$ and $W(G)$, respectively. The concept of interval edge-coloring of graphs was introduced by Asratian and Kamalian [3] in 1987. In [3], the authors proved that if $G \in \mathfrak{N}$, then $\chi'(G) = \Delta(G)$. Asratian and Kamalian also proved [3,4] that if a triangle-free graph G admits an interval t-coloring, then $t \leq |V(G)| - 1$.

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http://dx.doi.org/10.1016/j.dam.2017.04.005

In [18,19], Kamalian investigated interval colorings of complete bipartite graphs and trees. In particular, he proved that the complete bipartite graph $K_{m,n}$ has an interval t-coloring if and only if $m+n-\gcd(m,n) \leq t \leq m+n-1$, where $\gcd(m,n)$ is the greatest common divisor of m and n. In [25,29], Petrosyan, Khachatrian and Tananyan proved that the n-dimensional cube Q_n has an interval t-coloring if and only if $n \leq t \leq \frac{n(n+1)}{2}$. The problem of determining whether or not a given graph is interval colorable is NP-complete, even for regular [3] and bipartite [34] graphs. In some papers [3,4,11,12,16–22,24–30, 34], the problems of the existence, construction and estimating of the numerical parameters of interval colorings of graphs were investigated.

It is known that there are graphs that have no interval colorings. A smallest example is K_3. Since not all graphs admit an interval coloring, it is naturally to consider a measure of closeness for a graph to be interval colorable. In [13], Giaro, Kubale and Małafiejski introduced such a measure which is called deficiency of a graph (another measure was suggested in [32]). The *deficiency* $def(G)$ *of a graph* G is the minimum number of pendant edges whose attachment to G makes it interval colorable. The concept of deficiency of graphs can be also defined using proper edge-colorings. The *deficiency of a proper edge-coloring* α *at vertex* $v \in V(G)$, denoted by $def(v,\alpha)$, is the minimum number of integers which must be added to $S(v,\alpha)$ to form an interval, and the *deficiency* $def(G,\alpha)$ *of a proper edge-coloring* α *of* G is defined as the sum $\sum_{v\in V(G)} def(v,\alpha)$. In fact, $def(G) = \min_{\alpha} def(G,\alpha)$, where minimum is taken over all possible proper edge-colorings of G. Clearly, $def(G) = 0$ if and only if $G \in \mathfrak{N}$. The problem of determining the deficiency of a graph is NP-complete, even for regular and bipartite graphs [3,34,13]. In [13], Giaro, Kubale and Małafiejski obtained some results on the deficiency of bipartite graphs. In particular, they showed that there are bipartite graphs whose deficiency approaches the number of vertices. In [14], the same authors proved that if G is an r-regular graph with an odd number of vertices, then $def(G) \geq \frac{r}{2}$, and determined the deficiency of odd cycles, complete graphs, wheels and broken wheels. In [33], Schwartz investigated the deficiency of regular graphs. In particular, he obtained tight bounds on the deficiency of regular graphs and proved that there are regular graphs with high deficiency. Bouchard, Hertz and Desaulniers [9] derived some lower bounds for the deficiency of graphs and provided a tabu search algorithm for finding a proper edge-coloring with minimum deficiency of a graph. Recently, Borowiecka-Olszewska, Drgas-Burchardt and Hałuszczak [8] studied the deficiency of k-trees. In particular, they determined the deficiency of all k-trees with maximum degree at most $2k$, where $k \in \{2,3,4\}$. They also proved that the following lower bound for $def(G)$ holds: if G is a graph with an odd number of vertices, then $def(G) \geq \frac{2|E(G)|-(|V(G)|-1)\Delta(G)}{2}$. In the same paper, Borowiecka-Olszewska, Drgas-Burchardt and Hałuszczak posed the following conjecture on the deficiency of near-complete graphs: if $n \in \mathbb{N}$, then $def(K_{2n+1}-e) = n-1$.

For a graph G, the smallest and the largest values of t for which it has a proper t-edge-coloring α with deficiency $def(G,\alpha) = def(G)$ are denoted by $w_{def}(G)$ and $W_{def}(G)$, respectively. Clearly, if for a graph G, $def(G) = 0$, then $w_{def}(G) = w(G)$ and $W_{def}(G) = W(G)$. In this paper, we obtain some bounds on $w_{def}(G)$ and $W_{def}(G)$. We also determine the deficiency of certain graphs. In particular, we confirm the above-mentioned conjecture of Borowiecka-Olszewska, Drgas-Burchardt and Hałuszczak.

2. Notation, definitions and auxiliary results

We use standard notation C_n and K_n for the simple cycle and complete graph on n vertices, respectively. We also use standard notation $K_{m,n}$ and $K_{l,m,n}$ for the complete bipartite and tripartite graph, respectively, one part of which has m vertices, other part has n vertices and a third part has l vertices.

We denote by $\mathbb{Z}_+$ the set of nonnegative integers. For two positive integers a and b with $a \leq b$, we denote by $[a,b]$ the interval of integers $\{a, a+1, \ldots, b-1, b\}$. If $a > b$, then $[a,b] = \emptyset$.

Let A be a finite set of integers. The deficiency $def(A)$ of A is the number of integers between $\min A$ and $\max A$ not belonging to A. Clearly, $def(A) = \max A - \min A - |A| + 1$. A set A with $def(A) = 0$ is an interval. Note that if α is a proper edge-coloring of G and $v \in V(G)$, then $def(v,\alpha) = def(S(v,\alpha))$.

If α is a proper edge-coloring of a graph G and $v \in V(G)$, then the smallest and largest colors of the spectrum $S(v,\alpha)$ are denoted by $\underline{S}(v,\alpha)$ and $\overline{S}(v,\alpha)$, respectively. If α is a proper edge-coloring of a graph G and $V' \subseteq V(G)$, then we can define $S(V',\alpha)$ as follows: $S(V',\alpha) = \bigcup_{v\in V'} S(v,\alpha)$. The smallest and largest colors of $S(V',\alpha)$ are denoted by $\underline{S}(V',\alpha)$ and $\overline{S}(V',\alpha)$, respectively.

We will use the following five results.

Theorem 2.1 ([3,4]). *If* $G \in \mathfrak{N}$, *then* $\chi'(G) = \Delta(G)$. *Moreover, if* G *is a regular graph, then* $G \in \mathfrak{N}$ *if and only if* $\chi'(G) = \Delta(G)$.

Theorem 2.2 ([3,4]). *If* G *is a regular graph and* $G \in \mathfrak{N}$, *then for every* t, $w(G) \leq t \leq W(G)$, G *has an interval* t*-coloring.*

Theorem 2.3 ([3,4]). *If* G *is a triangle-free graph and* $G \in \mathfrak{N}$, *then* $W(G) \leq |V(G)| - 1$.

Theorem 2.4 ([5]). *If* G *is a planar graph and* $G \in \mathfrak{N}$, *then* $W(G) \leq \frac{11}{6}|V(G)|$.

Theorem 2.5 ([15]). *For any* $m,n \in \mathbb{N}$, $K_{1,m,n} \in \mathfrak{N}$ *if and only if* $\gcd(m+1,n+1) = 1$. *Moreover, if* $\gcd(m+1,n+1) = 1$, *then* $w(K_{1,m,n}) = m+n$.

We also need the following lemma on a special interval coloring of the complete bipartite graph $K_{p,p}$.