ARTICLE IN PRESS

Discrete Applied Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

q-cube enumerator polynomial of Fibonacci cubes

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ARTICLE INFO

Article history:
Received 21 July 2016
Received in revised form 7 April 2017
Accepted 19 April 2017
Available online xxxx

Keywords:
Hypercube
Fibonacci number
Fibonacci cube
Cube enumerator polynomial
q-analogue

ABSTRACT

We consider a q-analogue of the cube polynomial of Fibonacci cubes. These bivariate polynomials satisfy a recurrence relation similar to the standard one. They refine the count of the number of hypercubes of a given dimension in Fibonacci cubes by keeping track of the distances of the hypercubes to the all 0 vertex. For $q = 1$, they specialize to the standard cube polynomials.

We also investigate the divisibility properties of the q-analogues and show that the quotient polynomials for the appropriate indices have nonnegative integral polynomials in q as coefficients. These results have many corollaries which include expressions involving the q-analogues of the Fibonacci numbers themselves and their convolutions as they relate to hypercubes in Fibonacci cubes. Many of our developments can be viewed as refinements of enumerative results given by Klavžar and Mollard in (2012).

1. Introduction

A graph $G = (V, E)$ with vertex set V and edge set E can be used to represent an interconnection network. In this representation, V denotes the processors and E denotes the communication links between processors. The hypercube graph Q_n of dimension n is one of the basic model for interconnection networks. The vertices of Q_n are represented by all binary strings of length n and two vertices are adjacent if and only if they differ in exactly one position. The graph distance between two vertices of a graph is the length of the shortest path connecting these vertices. In Q_n, the graph distance between two vertices is given by the Hamming distance between the corresponding binary strings; this is the number of different bits of their binary representations. In Q_n, the weight of a vertex is defined as the number of ones in the corresponding string, that is, the Hamming weight of the string.

In [12], Fibonacci cubes were introduced as a new model of computation for interconnection networks. The Fibonacci cube Γ_n of dimension n is a subgraph of Q_n, where the vertices correspond to those without two consecutive 1s in their string representation. In other words, if we label the vertices of Γ_n $(n \geq 1)$ by using binary strings $b_1 b_2 \ldots b_n$ of length n, then the vertices of Γ_n have the property that $b_i b_{i+1} = 0$ for all $i \in \{1, 2, \ldots, n-1\}$. For convenience, Γ_0 is defined as Q_0, the graph with a single vertex and no edges.

In the literature, many interesting properties and applications of the Fibonacci cubes are presented. Their usage as interconnection networks and properties that are important in network design are given in [12,7]. In [13], the usage in theoretical chemistry and some results on the structure of Fibonacci cubes, including representations, recursive construction, hamiltonicity, the nature of the degree sequence and some enumeration results are presented. Many additional new properties of Fibonacci cubes are given in the literature, see for example [1,15,17,20]. Furthermore, the structure of the

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http://dx.doi.org/10.1016/j.dam.2017.04.026

hypercubes in Γ_n is studied in detail in the literature. The cube polynomial of Γ_n which is the starting point of this paper is studied in [14] and many interesting related results are obtained. The characterization of maximal induced hypercubes in Γ_n appears in [16]. Maximum number of disjoint subgraphs isomorphic to k-dimensional hypercube Q_k in Γ_n and their ratio to the number of vertices in Γ_n are considered in [8,18].

In this paper, we consider the q-analogue of the cube polynomial of the Fibonacci cubes. Many of our results are extensions of the work of Klavžar and Mollard as our q-cube polynomial $c_n(x; q)$ is a refinement of the cube polynomial $c_n(x)$ given in [14]. Furthermore, the q-analogue adds a geometric meaning to the polynomials; the $c_n(x; q)$ satisfy a simple recursion similar to the recursion for the cube polynomial $c_n(x)$ and have a combinatorial interpretation as enumerators of the hypercubes in Γ_n in which distance information of each hypercubes to the all 0 vertex is cataloged. For example,

$$c_2(x) = 3 + 2x$$

since Γ_2 contains three Q_0's and two Q_1's, whereas

$$c_2(x; q) = 1 + 2q + 2x$$

expresses the fact that two of the three Q_0 are at distance 1 from 00 and the other at distance 0; and both Q_1's are at distance 0 from 00 (i.e. they contain 00).

Certain divisibility properties of the cube polynomials were noted in [14]. Our results extend these divisibility properties and also includes information about the nature of the quotient polynomials. Interestingly, the quotients as polynomials in x have coefficients that are polynomials in q which have nonnegative integral coefficients themselves.

The distance information of the hypercubes in Γ_n maintained in $c_n(x; q)$ also has an interpretation in terms of the ranks when Γ_n is viewed as a subposet of the Boolean algebra Q_n, but this is not the emphasis of the present work.

The paper is organized as follows: In Section 2, we give some preliminaries. We present our q-cube enumerator polynomial in Section 3 and investigate divisibility properties in Section 4. In Section 5, we present additional results including the role of Fibonacci numbers and their q-analogues in the construction of $c_n(x; q)$, and a closed form expression for the simple q-analogue of the hypercube's own subcube enumerator.

2. Preliminaries

In this section, we present some notation and preliminary results related to Fibonacci cubes. We start with the description of a hypercube. An n-dimensional hypercube (or n-cube) Q_n is the simple graph with vertex set

$$V(Q_n) = \{v_1v_2\cdots v_n \mid v_i \in \{0, 1\},\ 1 \le i \le n\}.$$

The number of vertices in Q_n is 2^n and the number of these without two consecutive 1s is enumerated by the Fibonacci numbers. From this point of view, Fibonacci cube Γ_n can be considered as a subgraph of Q_n, obtained from Q_n by removing all vertices containing consecutive 1s. The vertex set of Γ_n can be shown as

$$V(\Gamma_n) = \{v_1v_2\cdots v_n \mid v_i \in \{0, 1\} \text{ with } v_iv_{i+1} = 0 \text{ for } 1 \le i < n\}. \tag{1}$$

By convention Γ_0 is defined as K_1. The number of vertices of the Fibonacci cube Γ_n is f_n, where $f_0 = 1, f_1 = 2$ and $f_n = f_{n-1} + f_{n-2}$ for $n \ge 2$. These are the Fibonacci numbers F_n shifted by 2: i.e. $f_n = F_{n+2}$ where $F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \ge 2$. In Fig. 1, we present the first 6 Fibonacci cubes with their vertices labeled with the corresponding binary strings in the hypercube graph. Since there is a close relationship between hypercubes and Fibonacci cubes, it is natural to consider the number of k-dimensional hypercubes in Γ_n in more detail. The enumerator of these subcubes in the Fibonacci cube Γ_n was considered in [14]. Here, we are considering a generalization of these polynomials (see, Section 3).

Fibonacci cubes have a useful decomposition, which is called the "fundamental decomposition" in [13]. For $n \ge 1$, the vertex set (1) of Γ_n can be partitioned into two disjoint subsets A_n and B_n as follows:

$$A_n = \{1v \mid v \in B_{n-1}\} \quad \text{and} \quad B_n = \{0v \mid v \in A_{n-1} \cup B_{n-1}\}$$

with $A_0 = \emptyset$ and $B_0 = \{\epsilon \mid \epsilon \text{ is the empty string}\}$. Note that for $n \ge 2$, we know that the label of any vertex in A_n must start with 10 by the definition of Γ_n. From this decomposition, one can see that A_n and B_n induce subgraphs of Γ_n isomorphic to Γ_{n-2} and Γ_{n-1}, respectively. We will show this fundamental decomposition for $n \ge 2$ as

$$\Gamma_n = 0\Gamma_{n-1} + 10\Gamma_{n-2}. \tag{2}$$

Note that there are edges between $0\Gamma_{n-1}$ and $10\Gamma_{n-2}$, namely a perfect matching between $00\Gamma_{n-2}$ and $10\Gamma_{n-2}$. We will use this property in the proof of Lemma 1.

In this paper, we consider the q-analogue of the Fibonacci numbers given by $F_0(q) = 0$, $F_1(q) = 1$, and

$$F_n(q) = F_{n-1}(q) + qF_{n-2}(q) \tag{3}$$

for $n \ge 2$. Note that for $q = 1$, $F_n(q)$ gives the Fibonacci numbers. Furthermore, this q-analogue is considered in [10] as Jacobsthal polynomial, and it is simpler than the standard one defined by

$$F'_n = F'_{n-1} + q^{n-2}F'_{n-2} \tag{4}$$

due to Schur, which was studied by Carlitz, Cigler, and others in the literature (see, for example [4–6,9]).

Please cite this article in press as: E. Saygı, Ö. Eğecioğlu, q-cube enumerator polynomial of Fibonacci cubes, Discrete Applied Mathematics (2017), http://dx.doi.org/10.1016/j.dam.2017.04.026