Discrete Applied Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Graphs with integer matching polynomial zeros

S. Akbari [a], P. Csikvári [b], A. Ghafari [a], S. Khalashi Ghezelahmad [c,*], M. Nahvi [a]

[a] *Department of Mathematical Sciences, Sharif University of Technology, Tehran, Iran*

[b] *MTA-ELTE Geometric and Algebraic Combinatorics Research Group & Alfréd Rényi Institute of Mathematics, Hungary*

[c] *Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran*

ARTICLE INFO

Article history:
Received 19 August 2016
Received in revised form 17 December 2016
Accepted 13 February 2017
Available online xxxx

Keywords:
Matching integral
Matching polynomial

ABSTRACT

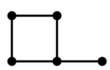
In this paper, we study graphs whose matching polynomials have only integer zeros. A graph is matching integral if the zeros of its matching polynomial are all integers. We characterize all matching integral traceable graphs. We show that apart from $K_7 \setminus (E(C_3) \cup E(C_4))$ there is no connected k-regular matching integral graph if $k \geq 2$. It is also shown that if G is a graph with a perfect matching, then its matching polynomial has a zero in the interval $(0, 1]$. Finally, we describe all claw-free matching integral graphs.

1. Introduction

All graphs we consider are finite, simple and undirected. Let G be a graph. We denote the edge set and the vertex set of G by $E(G)$ and $V(G)$, respectively. By *order* and *size* of G, we mean the number of vertices and the number of edges of G, respectively. The maximum degree of G is denoted by $\Delta(G)$ (or by Δ if G is clear from the context). The minimum degree of G is denoted by $\delta(G)$. In this paper, we denote the complete graph, the path and the cycle of order n, by K_n, P_n and C_n, respectively. The set of neighbors of a vertex v is denoted by $N(v)$. A *traceable graph*, is a graph with a Hamilton path. An r-*matching* in a graph G is a set of r pairwise non-incident edges. The number of r-matchings in G is denoted by $p(G, r)$. The *matching polynomial* of G is defined by

$$\mu(G, x) = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^r p(G, r) x^{n-2r},$$

where n is the order of G and $p(G, 0)$ is considered to be 1, see [7,9,8,10,11]. For instance the matching polynomial of the following graph



is $\mu(G, x) = x^5 - 5x^3 + 4x$.

By the definition of $\mu(G, x)$, we conclude that every graph of odd order has 0 as a matching root. Furthermore, if θ is a matching zero of a graph, then so is $-\theta$. We call a graph, *matching integral* if all zeros of its matching polynomial are integers. A graph is said to be *integral* if eigenvalues of its adjacency matrix consist entirely of integers. Since 1974, integral graphs

* Corresponding author.
E-mail addresses: s_akbari@sharif.edu (S. Akbari), peter.csikvari@gmail.com (P. Csikvári), ghafaribaghestani_a@mehr.sharif.edu (A. Ghafari), s.ghezelahmad@srbiau.ac.ir (S. Khalashi Ghezelahmad), nahvi_mina@mehr.sharif.edu (M. Nahvi).

http://dx.doi.org/10.1016/j.dam.2017.02.009
0166-218X/

have been extensively studied by several authors, for instance see [2,12]. It is worth mentioning that if T is a tree, then its characteristic polynomial and its matching polynomial are the same, see [6, Corollary 1.4, p. 21]. Integral trees (so matching integral trees) have been investigated in [16].

In Section 2, we characterize all traceable graphs which are matching integral. In Section 3, we study matching integral regular graphs and show that for $k \geq 2$ there is only one connected matching integral k-regular graph, namely $K_7 \setminus (E(C_3) \cup E(C_4))$. In order to establish our results, first we need the following theorems:

Theorem A *([13]). For any graph G, the zeros of $\mu(G, x)$ are all real. If $\Delta > 1$, then the zeros lie in the interval $(-2\sqrt{\Delta - 1}, 2\sqrt{\Delta - 1})$.*

Remark 1. Let G be a graph. Theorem A implies that if $\sqrt{\Delta - 1}$ is not an integer, then $\mu(G, x)$ contains at most $2\lfloor 2\sqrt{\Delta - 1}\rfloor + 1$ distinct integer zeros and if $\sqrt{\Delta - 1}$ is an integer, then $\mu(G, x)$ has at most $4\sqrt{\Delta - 1} - 1$ distinct integer zeros.

Theorem B *([6, Corollary 1.3, p. 97]). If G is a connected graph, then the largest zero of $\mu(G, x)$ has multiplicity 1. In other words, it is a simple zero.*

Let $t(G)$ be the number of vertices of a longest path in the graph G.

Theorem C *([6, Theorem 4.5, p. 107]).*

(a) *The maximum multiplicity of a zero of $\mu(G, x)$ is at most equal to the number of vertex-disjoint paths required to cover G.*
(b) *The number of distinct zeros of $\mu(G, x)$ is at least $t(G)$.*
(c) *In particular, if the graph G is traceable then all zeros of $\mu(G, x)$ are simple.*

Theorem D *([6]). If θ is a zero of $\mu(G, x)$ with multiplicity at least 2 then for any path P we have that θ is a zero of $\mu(G \setminus P, x)$, where $G \setminus P$ is the induced subgraph of G on the vertex set $V(G) \setminus V(P)$.*

Theorem D is not stated as a theorem in [6], but is used in the proof of Theorem 4.5 of Chapter 6 of [6]. Both Theorems C and D rely on the curious identity

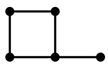
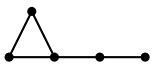
$$\mu'(G, x)^2 - \mu(G, x)\mu''(G, x) = \sum \mu(G \setminus P, x)^2,$$

where the sum is taken over all paths of G. For instance, if θ is a zero of $\mu(G, x)$ with multiplicity at least 2 then it is a zero of both $\mu(G, x)$ and $\mu'(G, x)$ so the left hand side is 0 at θ, but the right hand side is only 0 if all terms are 0.

2. Matching integral traceable graphs

In this section, we show that there are finitely many matching integral traceable graphs and characterize all of them. In fact, we will characterize those graphs whose matching polynomial has only simple integer zeros. By Theorem C we know that the matching polynomial of a traceable graph has only simple zeros. Hence this way we characterize matching integral traceable graphs.

Theorem 2.1. *Let G be a connected graph whose matching polynomial has only simple integer zeros. Then G is one of the following graphs: K_1, K_2, $K_7 \setminus (E(C_3) \cup E(C_4))$, G_1 or G_2, where*

G_1 :  G_2 : 

In particular, this is the list of matching integral traceable graphs.

Proof. Let n and m be the order and the size of G, respectively. It is enough to prove the first part of the theorem as the second part of the theorem indeed follows from the first one: since G is traceable, by Theorem C, the zeros of $\mu(G, x)$ are all distinct. Now, in order to prove the first part, we consider two cases:

Case 1. $n = 2k$, $k \geq 1$. Since G has even order and all zeros are simple, every zero of $\mu(G, x)$ is different from 0. Let $\theta_1, \ldots, \theta_k$ be the positive zeros of $\mu(G, x)$. Hence

$$\mu(G, x) = \prod_{i=1}^{k}(x^2 - \theta_i^2) = x^{2k} - (\theta_1^2 + \cdots + \theta_k^2)x^{2k-2} + \cdots + (-1)^k\theta_1^2 \cdots \theta_k^2.$$

We have

$$m = \sum_{i=1}^{k} \theta_i^2 \geq \sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6} = \frac{n}{12}\left(\frac{n}{2} + 1\right)(n+1).$$

Please cite this article in press as: S. Akbari, et al., Graphs with integer matching polynomial zeros, Discrete Applied Mathematics (2017), http://dx.doi.org/10.1016/j.dam.2017.02.009