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Zero forcing propagation time on oriented graphs

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ABSTRACT

Zero forcing is an iterative coloring procedure on a graph that starts by initially coloring vertices white and blue and then repeatedly applies the following rule: if any blue vertex has a unique (out-)neighbor that is colored white, then that neighbor is forced to change color from white to blue. An initial set of blue vertices that can force the entire graph to blue is called a zero forcing set. In this paper we consider the minimum number of iterations needed for this color change rule to color all of the vertices blue, also known as the propagation time, for oriented graphs. We produce oriented graphs with both high and low propagation times, consider the possible propagation times for the orientations of a fixed graph, and look at balancing the size of a zero forcing set and the propagation time.

1. Introduction

Given a directed graph with no loops or multiple arcs (i.e., a simple digraph), there are many possible processes that can be used to simulate information spreading. In the simplest model, each vertex can have two states, knowing or not knowing (using the colors blue and white, respectively), and then have a color change rule for changing a vertex from not knowing to knowing (i.e., changes from white to blue). For each possible color change rule there are a variety of questions including finding the minimum number of vertices that if initially colored blue will eventually change all the vertices blue, or finding the length of time it takes for a graph to become blue. The goal in this paper is to consider a particular color change rule, known as zero forcing, and to focus on the amount of time it takes to turn all the vertices blue, known as propagation time, on digraphs and specifically oriented graphs.

The zero forcing process on a simple digraph is based on an initial coloring of each vertex as blue or white and the repeated application of the following coloring rule: If a blue vertex has exactly one white out-neighbor, then that out-neighbor will change from white to blue. In terms of rumor spreading, this can be rephrased in the following way: "If I know a secret and all except one of my friends knows the same secret, then I will share that secret with my friend that doesn't know". The *zero forcing number* is the minimum number of vertices initially colored blue that can transform the entire graph to blue.

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The process of zero forcing was introduced originally for (simple, undirected) graphs by mathematical physicists in [8] and by combinatorial matrix theorists in [2]. A set of vertices that can color the entire graph blue can also control the quantum system [7]. The maximum nullity of a graph is the maximum of the nullities of symmetric matrices whose nonzero off-diagonal pattern is described by the edges of the graph. The zero forcing number of the graph is an upper bound on nullity of such a matrix [2] (in fact, the term "zero forcing" comes from forcing zeros in a null vector of such a matrix).

In general, the zero forcing number can be determined computationally but the problem of computing the zero forcing number is NP-hard [1]. However, the zero forcing number has been determined for several families of graphs (see, for example, the recent survey by Fallat and Hogben [12]), and bounds have been established in some cases [3,13,17]. There has also been additional work done on applications to control of quantum systems [7] and structural control [19]. Variants such as the k-forcing number [3] and fractional zero forcing [16] have recently been introduced and studied. Zero forcing was extended to digraphs in Barioli et al. [4]. Zero forcing for simple digraphs was studied in [5,14].

Most of the focus of the literature has been on the determination of the zero forcing number. However, given a minimum set of vertices initially colored blue that can transform the entire graph to blue, another natural question to examine is the amount of time it takes to turn all of the vertices blue (i.e., the propagation time). This study was initiated for undirected graphs in Hogben et al. [15] where extremal configurations were determined (i.e., n-vertex graphs that propagate as quickly or as slowly as possible) and in Chilakamarri et al. [11] where propagation time (there called *iteration index*) was computed for some families of graphs. This paper expands the study of propagation time to oriented graphs. In particular, there are some subtle and important distinctions between undirected graphs and oriented graphs.

In the remainder of the introduction we introduce the notation and give precise terminology. In Section 2 we show that the propagation time is not affected when the direction of each arc in a simple digraph is reversed. In Sections 3 and 4 we consider orientations of graphs that have low and high propagation times, respectively. For a given graph G there are many possible orientations and this gives rise to the following problem: *For a given graph G, find the propagation time of $\overrightarrow{G}$ as $\overrightarrow{G}$ ranges over all possible orientations of G.* In Section 5 we consider such orientation propagation intervals. In Section 6 we consider what happens when we balance the size of the zero forcing set with the propagation time; in particular we show that, unlike in simple graphs, we cannot always obtain significant savings in the sum by increasing the size of the zero forcing set. Finally, in Section 7 we discuss other approaches to propagation time for oriented graphs, including computation of propagation time for a given oriented graph and consideration of variation of propagation times across more than one minimum zero forcing set.

1.1. Terminology and definitions

A *simple graph* (respectively, *simple digraph*) is a finite undirected (respectively, directed) graph that does not allow loops or more than one copy of one edge or arc; a simple digraph does allow *double* arcs, i.e., both the arcs (u, v) and (v, u). We use $G = (V(G), E(G))$ to denote a simple graph and $\Gamma = (V(\Gamma), E(\Gamma))$ to denote a simple digraph, where V and E are the vertex and edge (or arc) sets, respectively. Furthermore, we let $|G| = |V(G)|$ denote the number of vertices of G, and similar notation is used for digraphs. An *oriented graph* is a simple digraph in which there are no double arcs, i.e., if (u, v) is an arc in Γ then (v, u) is not an arc in Γ. For a simple graph G, we also let $\overrightarrow{G}$ denote an orientation of G,[1] i.e., $\overrightarrow{G}$ is an oriented graph such that ignoring the orientations of the arcs gives the graph G.

For a digraph Γ having $u, v \in V(\Gamma)$ and $(u, v) \in E(\Gamma)$, we say that v is an *out-neighbor* of u and that u is an *in-neighbor* of v. The set of all in-neighbors of v is denoted by $N^-(v)$ and the cardinality of $N^-(v)$ is the *in-degree* of v, denoted by $\deg^-(v)$. Similarly, the set of all out-neighbors of v is $N^+(v)$ and the cardinality of $N^+(v)$ is the *out-degree*, denoted by $\deg^+(v)$.

For a simple digraph Γ, the zero forcing propagation process can be described as follows. Let $B \subseteq V(\Gamma)$, let $B^{(0)} := B$ and iteratively define $B^{(t+1)}$ as the set of vertices w where for some $v \in \bigcup_{i=0}^{t} B^{(i)}$ we have that w is the unique out-neighbor of v that is not in $\bigcup_{i=0}^{t} B^{(i)}$. Here $B^{(0)}$ represents the initial set of vertices colored blue, and at each stage we color as many vertices blue as possible (i.e., we apply the coloring rule simultaneously to all vertices). We say a set B is a *zero forcing set* if $\bigcup_{i=0}^{t} B^{(i)} = V(\Gamma)$ for some t. Further, the propagation time of B, denoted by $\mathrm{pt}(\Gamma, B)$, is the minimum t so that $\bigcup_{i=0}^{t} B^{(i)} = V(\Gamma)$ (i.e., the minimum amount of time needed for B to color the entire graph blue).

One way to achieve fast propagation is to simply let $B = V(\Gamma)$, and be done at time 0. However, we are primarily interested in the propagation time of a zero forcing set B of minimum cardinality, called a *minimum zero forcing set*. In particular, for a simple digraph Γ, we let $Z(\Gamma)$ denote the cardinality of a minimum zero forcing set for Γ. We then define propagation time as follows:

$$\mathrm{pt}(\Gamma) = \min\{\mathrm{pt}(\Gamma, B) : B \text{ is a minimum zero forcing set}\}.$$

Example 1.1. Consider the oriented graph $\overrightarrow{G}$ shown in Fig. 1. Since the vertices c, d and f are not the out-neighbors of any vertices, they cannot be changed to blue by the coloring rule. Therefore these three vertices *must* be in every zero forcing set of $\overrightarrow{G}$. We now show that these three vertices form a zero forcing set (and in particular this is the unique minimum cardinality zero forcing set), allowing us to conclude $Z(\overrightarrow{G}) = 3$. Suppose that $B^{(0)} = \{c, d, f\}$ and mark these vertices by

[1] For visual simplicity, the arrow is only over the main symbol, e.g., an orientation of K_n is denoted by $\overrightarrow{K}_n$ rather than $\overrightarrow{K_n}$.