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Extremal anti-forcing numbers of perfect matchings of graphs

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ABSTRACT

The anti-forcing number of a perfect matching M of a graph G is the minimal number of edges not in M whose removal to make M as a unique perfect matching of the resulting graph. The set of anti-forcing numbers of all perfect matchings of G is the anti-forcing spectrum of G. In this paper, we characterize the plane elementary bipartite graph whose minimum anti-forcing number is one. We show that the maximum anti-forcing number of a graph is at most its cyclomatic number. In particular, we characterize the graphs with the maximum anti-forcing number achieving the upper bound, such extremal graphs are a class of plane bipartite graphs. Finally, we determine the anti-forcing spectrum of an even polygonal chain in linear time.

1. Introduction

We only consider finite and simple graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A *perfect matching* or 1-factor of G is a set of disjoint edges which covers all vertices of G. A perfect matching of a graph coincides with a Kekulé structure in organic chemistry and a dimer in statistic physics.

The concept of "forcing" has been used in many research fields in graph theory and combinatorics [3,17]. It appeared first in a perfect matching M of a graph G due to Harary et al. [9]: If a subset S of M is not contained in other perfect matchings of G, then we say S *forces* the perfect matching M, in other words, S is called a *forcing set* of M. The minimum cardinality over all forcing sets of M is called the *forcing number* of M. The roots of those concepts can be found in an earlier chemical literature due to Randić and Klein [12], under the name of the *innate degree of freedom* of a Kekulé structure, which plays an important role in the resonance theory in chemistry. In 1990s Zhang and Li [26] and Hansen and Zheng [8] determined independently the hexagonal systems with a forcing edge. Afterwards Zhang and Zhang [28] characterized plane elementary bipartite graphs with a forcing edge. For more researches on matching forcing problems, see [1,2,10,11,18,19,22,27,25].

Vukičević and Trinajstić [20] introduced the *anti-forcing number* of a graph G as the smallest number of edges whose removal results in a subgraph with a unique perfect matching, denoted by $af(G)$. So a graph G has a unique perfect matching if and only if its anti-forcing number is zero. An edge e of a graph G is called an *anti-forcing edge* if $G-e$ has a unique perfect matching. As early as 1997 Li [14] showed that the hexagonal systems with an anti-forcing edge (under the name "forcing single edge") are truncated parallelograms. Deng [4] gave a linear time algorithm to compute the anti-forcing number of benzenoid chains. Yang et al. [23] showed that a fullerene has the anti-forcing number at least four. For other works, see [21,5,24].

Recently, Lei et al. [13] defined the anti-forcing number of a perfect matching M of a graph G as the minimal number of edges not in M whose removal to make M as a single perfect matching of the resulting graph, denoted by $af(G, M)$. By this

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definition, $af(G)$ is the smallest anti-forcing number over all perfect matchings of G, called the *minimum anti-forcing number* of G. Also let $Af(G)$ denote the largest anti-forcing number over all perfect matchings of G, called the *maximum anti-forcing number* of G. They also showed that the maximum anti-forcing number of a hexagonal system equals its Fries number. The present authors [6,7] considered the *anti-forcing spectrum* $\mathrm{Spec}_{af}(G)$ as the set of anti-forcing numbers of perfect matchings in G.

Let M be a perfect matching of a graph G. A subset $S \subseteq E(G) \setminus M$ is called an *anti-forcing set* of M if M is the unique perfect matching of $G - S$. A cycle C of G is called an *M-alternating cycle* if the edges of C appear alternately in M and $E(G) \setminus M$. If C is an M-alternating cycle of G, then the symmetric difference $M \triangle C := (M - C) \cup (C - M)$ is another perfect matching of G. In this case a cycle is always regarded as its edge set.

Theorem 1.1 ([13]). *An edge set S of a graph G is an anti-forcing set of a perfect matching M of G if and only if S contains at least one edge of every M-alternating cycle of G.*

Let M be a perfect matching of a graph G. A set $\mathcal{A}$ of M-alternating cycles of G is called a *compatible M-alternating set* if any two members of $\mathcal{A}$ either are disjoint or intersect only at edges in M. Let $c'(M)$ denote the cardinality of a maximum compatible M-alternating set of G. By Theorem 1.1, we have $af(G, M) \geq c'(M)$. For plane bipartite graphs, the equality holds.

Theorem 1.2 ([13]). *Let G be a plane bipartite graph with a perfect matching M. Then $af(G, M) = c'(M)$.*

Throughout this paper all the bipartite graphs are given a proper black and white coloring: any two adjacent vertices receive different colors. An edge of a graph G is *allowed* if it belongs to a perfect matching of G and *forbidden* otherwise. G is said to be *elementary* if all its allowed edges form a connected subgraph of G. It is well-known that a connected bipartite graph is elementary if and only if each edge is allowed [16].

An elementary bipartite graph has the so-called "*bipartite ear decomposition*". Let x be an edge. Join the end vertices of x by a path P_1 of odd length (the so-called "first ear"). We proceed inductively to build a sequence of bipartite graphs as follows: If $G_{r-1} = x + P_1 + P_2 + \cdots + P_{r-1}$ has already been constructed, add the rth ear P_r (a path of odd length) by joining any two vertices in different colors of G_{r-1} such that P_r has no other vertices in common with G_{r-1}. The decomposition $G_r = x + P_1 + P_2 + \cdots + P_r$ will be called a bipartite ear decomposition of G_r.

Theorem 1.3 ([15]). *A bipartite graph is elementary if and only if it has a bipartite ear decomposition.*

A bipartite ear decomposition $G = x+P_1+P_2+\cdots+P_r$ can be represented by a sequence of graphs $(G_0, G_1, \ldots, G_r(= G))$, where $G_0 = x$ and $G_i = G_{i-1} + P_i$ for $1 \leq i \leq r$. We can see that the number of ears equals $|E(G)| - |V(G)| + 1$, i.e., the *cyclomatic number* of G, denoted by $r(G)$.

A bipartite ear decomposition $(G_1(= x + P_1), \ldots, G_r(= G))$ of a plane elementary bipartite graph G is called a *reducible face decomposition* if G_1 is the boundary of an interior face of G and the ith ear P_i lies in the exterior of G_{i-1} such that P_i and the part of the periphery of G_{i-1} bound an interior face of G for all $2 \leq i \leq r$.

Theorem 1.4 ([28]). *Let G be a plane bipartite graph other than K_2. Then G is elementary if and only if G has a reducible face decomposition starting with the boundary of any interior face of G.*

In Section 2, we characterize the plane elementary bipartite graphs with anti-forcing edges by using reducible face decomposition. In Section 3, we show that the maximum anti-forcing number of a connected graph with a perfect matching is at most its cyclomatic number. In particular we characterize the graphs with the maximum anti-forcing number achieving this cyclomatic number in terms of bipartite ear decomposition. We shall see that such extremal graphs are a special type of plane bipartite graphs, and have a unique perfect matching whose anti-forcing number is maximum. In Section 4, we show that an even polygonal chain including benzenoid chain has the continuous anti-forcing spectrum. So we can determine the anti-forcing spectrum by designing linear algorithms to compute the minimum and maximum anti-forcing numbers of an even polygonal chain.

2. Anti-forcing edge

The *Z-transformation graph* $Z(G)$ of a plane bipartite graph G is defined as the graph whose vertices represent the perfect matchings of G where two vertices are adjacent if and only if the symmetric difference of the corresponding two perfect matchings just forms the boundary of an interior face of G. A face of G is said to be *resonance* if its boundary is an M-alternating cycle with respect to a perfect matching M of G. By using reducible face decomposition, Zhang and Zhang [28] described those plane elementary bipartite graphs whose Z-transformation graphs have a vertex of degree one and characterized the plane elementary bipartite graphs with a forcing edge.

Theorem 2.1 ([28]). *A plane elementary bipartite graph G has a forcing edge if and only if G has a perfect matching M such that G has exactly two M-resonance faces (the exterior face is allowed) and their boundaries are intersecting. Further each common edge in M on the two M-resonance faces is a forcing edge of G.*

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