Discrete Applied Mathematics 224 (2017) 91–105

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

On well-covered pentagonalizations of the plane

Arthur S. Finbow [a,*], Bert L. Hartnell [a], Michael D. Plummer [b]

[a] Department of Mathematics and Computing Science, Saint Mary's University, Halifax, Canada B3H 3C3
[b] Department of Mathematics, Vanderbilt University, Nashville, TN 37240, United States

ARTICLE INFO

Article history:
Received 20 June 2014
Received in revised form 17 February 2017
Accepted 19 February 2017
Available online 5 April 2017

Keywords:
Independent set
Well-covered
Plane pentagonalization

ABSTRACT

A graph G is said to be *well-covered* if every maximal independent set of vertices has the same cardinality. A planar (simple) graph in which each face is bounded by a pentagon is called a (planar) *pentagonalization*.

In the present paper we investigate those planar pentagonalizations which are well-covered.

1. Introduction

One of the cornerstones of modern day theoretical computing is the idea of an *NP*-complete problem due to Cook [6]. To put it simply, this problem class consists of a collection of problems each of which can be transformed to any other via a polynomial algorithm. The famous unsettled problem here is whether or not there is a polynomial algorithm to solve any one of these problems.

One of the first problems shown to be *NP*-complete (cf. Karp [15]) was that of finding a largest independent set of vertices in an arbitrary graph. For certain interesting sub-families of graphs the problem has been shown to be polynomially solvable. One such special class is the *claw-free* graphs. (See Minty [16] and Sbihi [20].)

Another class of graphs for which the independent set problem is solvable – in fact, trivially so – is the class of *well-covered* graphs. A graph is said to be *well-covered* [17] if every maximal independent set of vertices is maximum. Or in other words, every maximal independent set of vertices has the same cardinality.

But how does one recognize the class of well-covered graphs? No polynomial algorithm is known for solving this recognition problem. See, however [5,19,21,22].

On the other hand, various subclasses of well-covered graphs have been shown to be polynomially recognizable (cf. [1–4,7,8,13,9,11,12].) For more comprehensive surveys of well-covered graphs, see Plummer [18] and more recently, Hartnell [14].

A widely studied class of graphs are those which are maximal planar and which are commonly called (planar) *triangulations*. Clearly, any triangulation (larger than a single triangle) must have vertex connectivity 3, 4 or 5. In an earlier paper [9] it was shown that there is no 5-connected planar well-covered triangulation. Papers [10,11] culminated in the characterization of those 4-connected planar triangulations which are well-covered. There are precisely four of these and they are labeled R_6, R_7, R_8, and R_{12} in Fig. 1.1.

Finally, in a very recent paper (cf. [12]) the determination of the entire class of well-covered planar triangulations was completed.

* Corresponding author.
E-mail addresses: art.finbow@stmarys.ca (A.S. Finbow), bert.hartnell@stmarys.ca (B.L. Hartnell), michael.d.plummer@vanderbilt.edu (M.D. Plummer).

http://dx.doi.org/10.1016/j.dam.2017.02.016

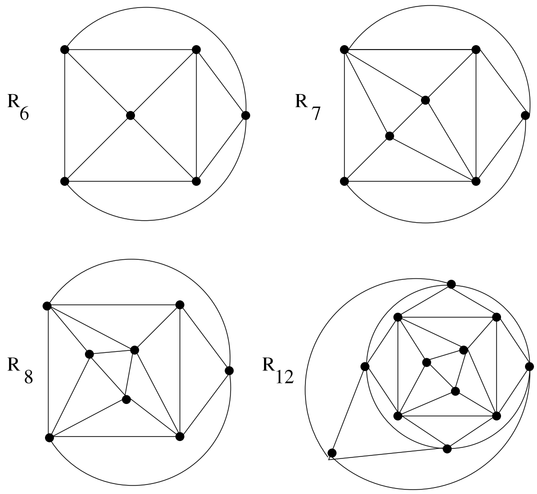

Fig. 1.1. The four 4-connected planar well-covered triangulations.

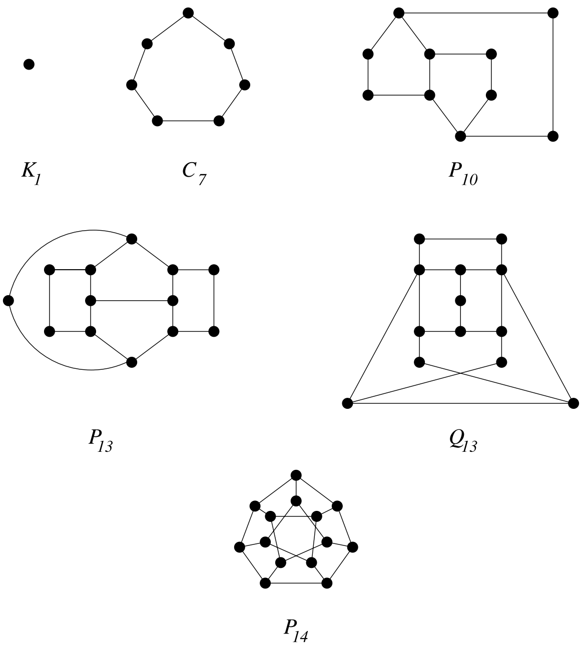

Fig. 1.2. The six orphans.

The class of all well-covered planar quadrangulations have been characterized as well (cf. [13]). In the present paper we investigate the well-covered members of the one remaining face-regular class of plane graphs, namely, the pentagonalizations.

Clearly, the girth (or length of a shortest cycle) of a pentagonalization cannot exceed 5. But equality here need not hold for a pentagonalization, even if it is well-covered. If the girth of a well-covered pentagonalization is exactly 5, the structure of the graph is known via a characterization due to [7] which we now introduce and which we shall subsequently need.

A 5-cycle C in a graph G is said to be *basic* if C does not contain two adjacent vertices of degree 3 or more in G. A graph G belongs to the family $\mathcal{PC}$ if $V(G)$ can be partitioned into two subsets P and C, where P consists of those vertices incident with pendant edges and the pendant edges form a perfect matching of P whereas C consists of the vertices of all basic 5-cycles in G and these basic 5-cycles partition the vertex set C.

Theorem 1.1 (*[7]*)**.** *Let G be a well-covered graph having girth at least* 5*. Then if G is in the class $\mathcal{PC}$, G is well-covered. On the other hand, if G is a connected well-covered graph of girth $\geq$ 5, then G either belongs to the class $\mathcal{PC}$ or is isomorphic to one of the six exceptional graphs K_1, C_7, P_{10}, P_{13}, Q_{13} or P_{14} shown in Fig. 1.2. We sometimes refer to these six exceptional graphs as orphans.*