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Proximity, remoteness and girth in graphs

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ABSTRACT

The proximity π of a graph G is the minimum average distance from a vertex of G to all others. Similarly, the remoteness of G is the maximum average distance from a vertex to all others. The girth g of a graph G is the length of its smallest cycle. In this paper, we provide and prove sharp lower and upper bounds, in terms of the order n of G, on the difference, the sum, the ratio and the product of the proximity and the girth. We do the same for the remoteness and the girth, except for the lower bound on ρ/g, which is already known.

1. Introduction

Models involving paths, distances and location on graphs are much studied in operations research and mathematics. Models from operations research (see e.g. [8,15]) usually use weighted graphs to describe some well-defined class of problems or some specific applications. Models from mathematics most often consider unweighted graphs and relations between graph invariants, *i.e.*, numerical quantities whose values do not depend on the labeling of edges or vertices. In this paper, we continue the study of two recently introduced [1,5] graph invariants: proximity and remoteness, defined as the minimum and maximum of the average distance from a vertex to all others. These two invariants are compared with the girth *i.e.*, the length of the shortest cycle in the graph under consideration.

Let $G = (V, E)$ denote a simple and connected graph, with vertex set V and edge set E, containing $n = |V|$ vertices and $m = |E|$ edges. All the graphs considered in the present paper are finite, simple and connected. We also assume that $m \geq n$, *i.e.*, each of the considered graphs contains at least one cycle, and by the way, the trees are excluded from our study. The distance between two vertices u and v in G, denoted by $d(u, v)$, is the length of a shortest path between u and v. The average distance between all pairs of vertices in G is denoted by $\bar{l}$. The eccentricity $e(v)$ of a vertex v in G is the largest distance from v to another vertex of G. The minimum eccentricity in G, denoted by r, is the radius of G. The maximum eccentricity of G, denoted by D, is the diameter of G. The average eccentricity of G is denoted ecc. That is

$$r = \min_{v \in V} e(v), \qquad D = \max_{v \in V} e(v) \quad \text{and} \quad ecc = \frac{1}{n} \sum_{v \in V} e(v).$$

The *girth* g of the graph G is the length of its smallest cycle. The *proximity* π of G is the minimum average distance from a vertex of G to all others. Similarly, the *remoteness* of G is the maximum average distance from a vertex to all others. The two

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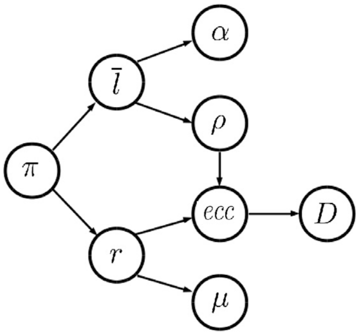

Fig. 1. Summary of relations proved in [5].

last concepts were recently introduced in [1,3]. They are close to the concept of transmission $t(v)$ of a vertex v, which is the sum of the distances from v to all others. That is, if we denote $\tilde{t}(v)$ the average distance from a vertex v to all other vertices in G, we have

$$\pi = \min_{v\in V} \tilde{t}(v) = \min_{v\in V} \frac{t(v)}{n-1} \quad \text{and} \quad \rho = \max_{v\in V} \tilde{t}(v) = \max_{v\in V} \frac{t(v)}{n-1}.$$

The transmission of a vertex is also known as the *distance* of a vertex [14] and the minimum distance (transmission) of a vertex is studied in [24]. A notion very close to the average distance from a vertex is the *vertex deviation* introduced by Zelinka [32] as

$$m_1(v) = \frac{1}{n} \sum_{u\in V} d(u, v) = \frac{t(v)}{n}.$$

The vector composed of the vertex transmissions in a graph was first introduced by Harary [18] in 1959, under the name *the status* of a graph, as a measure of the "weights" of individuals in social networks. The same vector was called *the distance degree sequence* by Bloom, Kennedy and Quintas [9]. It was used to tackle the problem of graph isomorphism. Randić [25] conjectured that two graphs are isomorphic if and only if they have the same distance degree sequence. The conjecture was refuted by several authors such as Slater [29], Buckley and Harary [10], and Entringer, Jackson and Snyder [14]. The transmission of a graph was also introduced by Sabidussi [26] in 1966 as a measure of *centrality* in social networks. The notion of centrality is widely used in different branches of sciences (see for example [22] and the references therein) such as transportation-network theory, communication network theory, electrical circuits theory, psychology, sociology, geography, game theory and computer science. Notions closely related to that of the distance from a vertex are those of *a center* and *a centroid* introduced by Jordan [21] in 1869. For mathematical properties of these two concepts see the survey, as well as the references therein, [30]. In 1964, Hakimi [17] used for the first time the sum of distances in solving facility location problems. In fact, Hakimi [17] considered two problems, subsequently considered in many works: the first problem was to determine a vertex $u \in V$ so as to minimize $\max_{v\in V}\{d(u, v) : u \in V\}$, *i.e.*, the center of a graph; and the second problem is to determine a vertex $u \in V$ so as to minimize the sum of distances from u, *i.e.*, the centroid. Interpretations of these problems can be found, for instance, in [16]. In view of the interest of the transmission vector in different domains of sciences, it is natural to study the properties of its extremal values themselves, and among the set of graph parameters. The study of proximity and remoteness, since closely related to respectively the minimum and maximum values of the transmissions, appears to be convenient, specially, with other metric invariants, such as the diameter, radius, average eccentricity and average distance. Indeed, it follows from the definitions that

$$\pi \le r \le ecc \le D, \qquad \pi \le \bar{l} \le \rho \le D \quad \text{and} \quad \bar{l} = \frac{1}{n(n-1)} \sum_{v\in V} t(v).$$

Since their introduction in [1,3], the proximity and the remoteness attracted the attention of several authors.

In [5], π and ρ were compared with r, D, ecc and $\bar{l}$, as well as with the independence number α and the matching number μ. The results proved in [5] are summarized in Fig. 1 where an arrow from an invariant a to an invariant b means that $a \le b$.

In [4], the authors proved Nordhaus–Gaddum type inequalities for both proximity π and remoteness ρ. They also characterized the extremal graphs corresponding to those related to those inequalities. In [28], Sedlar et al. the authors proved two AutoGraphiX (a software devoted to conjecture making in graph theory, see [1–3,11,12]) conjectures involving remoteness ρ, vertex connectivity (the minimum number of vertices whose removal disconnects G or reduces it to a single vertex) and algebraic connectivity (the second smallest Laplacian eigenvalue of G). An upper bound on the difference $ecc - \pi$, first conjectured using AutoGraphiX, was proved and the corresponding extremal graphs were characterized by Ma, Wu and Zhang in [23]. The paper [27], by Sedlar, was devoted to the study of three AutoGraphiX conjectures involving proximity and remoteness: one conjecture was proved and partial results were proved regarding the two others. The extremal graphs