Discrete Applied Mathematics 222 (2017) 76–88

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

# Restraints permitting the largest number of colourings

Jason Brown, Aysel Erey *

*Department of Mathematics and Statistics, Dalhousie University, Halifax, Nova Scotia, Canada B3H 3J5*

ARTICLE INFO

*Article history:*
Received 31 July 2015
Received in revised form 8 January 2017
Accepted 23 January 2017
Available online 13 February 2017
Communicated by E. Boros

*Keywords:*
Graph colouring
Restraint
Chromatic polynomial
Restrained chromatic polynomial
Bipartite graph

ABSTRACT

A *restraint* $r$ on $G$ is a function which assigns each vertex $v$ of $G$ a finite set of forbidden colours $r(v)$. A proper colouring $c$ of $G$ is said to be *permitted by the restraint* $r$ if $c(v) \notin r(v)$ for every vertex $v$ of $G$. A restraint $r$ on a graph $G$ with $n$ vertices is called a $k$*-restraint* if $\mid r(v) \mid= k$ and $r(v) \subseteq \{1, 2, \ldots, kn\}$ for every vertex $v$ of $G$. In this article we discuss the following problem: among all $k$-restraints $r$ on $G$, which restraints permit the largest number of $x$-colourings for all large enough $x$? We determine such extremal restraints for all bipartite graphs.



## 1. Introduction

In a number of applications of graph colourings, constraints on the colour sets naturally play a role. For example, when one sequentially colours the vertices of a graph under a variety of algorithms, lists of forbidden colours dynamically grow at each vertex as neighbours are coloured. In scheduling and timetable problems, individual preferences may constrain the allowable colours at each vertex (cf. [9]). There is the well-established and well-studied problem (see, for example, [1,4], Section 9.2 and [13]) of *list colourings*, where one has available at each vertex $v$ a list $L(v)$ of possible colours, which is equivalent to the remaining colours being forbidden at the node.

In all these applications, for each vertex $v$ we have a finite list of *forbidden* colours $r(v) \subset \mathbb{N}$, and we call the function $r$ a *restraint* on the graph $G$; the goal is to colour the graph subject to the restraint placed on the vertex set. More specifically, a proper $x$-colouring $c$ of $G$ is *permitted* by restraint $r$ if $c(v) \notin r(v)$ for all vertices $v$ of $G$. This notion is equivalent to a list colouring problem, where, if $C$ is the set of colours available, then the list of colours available at vertex $v$ is $L(v) = C - r(v)$. Conversely, we observe that if each vertex $v$ of a graph $G$ has a list of available colours $L(v)$, and

$$L = \bigcup_{v\in V(G)} L(v) \subseteq [N],$$

then setting $r(v) = [N]-L(v)$, we see that $G$ is list colourable with respect to the lists $L(v)$ if and only if $G$ has an $N$-colouring permitted by $r$.

The key difference is that in list colourings, the colour sets at each vertex are fixed, while we wish to allow the total possible number of colours to grow. In specific applications (such as have lists of unavailable time slots for each vertex in a scheduling problem but additional time slots can be added in order to meet the constraints), the notion of restraints may be more flexible than trying to model the problem as a list colouring.

* Corresponding author.
*E-mail address:* aysel.erey@gmail.com (A. Erey).

http://dx.doi.org/10.1016/j.dam.2017.01.022

A question of interest is whether there is a proper colouring that is permitted by a specific restraint. The ability to find, for an $x$-chromatic graph $G$ and for all non-constant restraints $r : V(G) \to \{1, 2, \ldots, x\}$, an $x$-colouring permitted by $r$ has been used in the construction of critical graphs (with respect to colourings) [12] and in the study of some other related concepts [3,10]. Our aim in this paper is to more fully investigate the *number* of colourings permitted by a given restraint.

To begin, we shall need a few definitions. Let $G$ be a graph on $n$ vertices. A *proper x-colouring* of $G$ is a function $f : V(G) \to \{1, 2, \ldots, x\}$ such that $f(u) \neq f(v)$ for every $uv \in E(G)$. We say that $r$ is a *k-restraint* on $G$ if $|r(u)| = k$ and $r(u) \subseteq \{1, 2, \ldots, kn\}$ for every $u \in V(G)$. If $k = 1$ (that is, we forbid exactly one colour at each vertex) we omit $k$ from the notation and use the word *simple* when discussing such restraints. If the vertices of $G$ are ordered as $v_1, v_2 \ldots v_n$, then we usually write $r$ in the form $[r(v_1), r(v_2) \ldots, r(v_n)]$, and when drawing a graph, we label each vertex with its list of restrained colours.

Given a restraint $r$ on a graph $G$, the *restrained chromatic polynomial* of $G$ with respect to $r$, denoted by $\pi_r(G, x)$, is defined as the number of $x$-colourings permitted by restraint $r$ [2]. Note that this function extends the definition of chromatic polynomial, $\pi(G, x)$ because if $r(v) = \emptyset$ for every vertex $v$, then $\pi_r(G, x) = \pi(G, x)$. Furthermore, it turns out that $\pi_r(G, x)$ is a polynomial function of $x$ when $x$ is large enough [2].

Our focus will be on the following question:

**Question 1.1.** Given a graph $G$ and a natural number $k$, among all $k$-restraints on $G$ which restraints permit the largest/smallest number of $x$-colourings for all large enough $x$?

Since $\pi_r(G, x)$ is a polynomial function of $x$, it is clear that such extremal restraints always exist for all graphs $G$. Let $R_{\max}(G, k)$ (resp. $R_{\min}(G, k)$) be the set of extremal $k$-restraints on $G$ permitting the largest (resp. smallest) number of colourings for sufficiently large number of colours.

In this article, we first give a complete answer to the minimization part of this question, by determining $R_{\min}(G, k)$ for all graphs $G$ (Corollary 3.4). (We remark that the results of Donner [6], Thomassen [11] and Wang et al. [14] on list colourings can also be used to derive this result.) We then turn our attention to the more difficult maximization problem. We give two necessary conditions for a restraint to be in $R_{\max}(G, k)$ for every graph $G$ (Theorem 4.5), and we show that these necessary conditions are sufficient to determine $R_{\max}(G, k)$ when $G$ is a bipartite graph (Corollary 4.8).

## 2. Preliminaries

Similar to the chromatic polynomial, the restrained chromatic polynomial also satisfies an edge deletion–contraction formula. Recall that $G \cdot uv$ is the graph formed from $G$ by contracting edge $uv$, that is, by identifying the vertices $u$ and $v$ (and taking the underlying simple graph).

**Lemma 2.1** (*Edge Deletion–Contraction Formula [2]*). *Let $r$ be any restraint on $G$, and $uv \in E(G)$. Suppose that $u$ and $v$ are replaced by $w$ in the contraction $G \cdot uv$. Then*

$$\pi_r(G, x) = \pi_r(G - uv, x) - \pi_{r_{uv}}(G \cdot uv, x)$$

*where*

$$r_{uv}(a) = \begin{cases} r(a) & \text{if } a \neq w \\ r(u) \cup r(v) & \text{if } a = w \end{cases}$$

*for each $a \in V(G \cdot uv)$.*

Given a restraint function $r$ on a graph $G$, let $M_{G,r}$ be the maximum value in $\bigcup_{v \in V(G)} r(v)$ if the set is nonempty and 0 otherwise. By using Lemma 2.1, it is easy to see that the following holds.

**Theorem 2.2** (*[2]*). *Let $G$ be a graph of order $n$ and $r$ be a restraint on $G$. Then for all $x \geq M_{G,r}$, the function $\pi_r(G, x)$ is a monic polynomial of degree $n$ with integer coefficients that alternate in sign.*

Let $A = [x_1, \ldots, x_n]$ be a sequence of variables. Then recall that for $i \in \{0, \ldots, n\}$, the $i$th elementary symmetric function on $A$ is equal to

$$S_i(A) = \sum_{1 \leq k_1 < \cdots < k_i \leq n} x_{k_1} \ldots x_{k_i}.$$

**Proposition 2.3.** *Let $r$ be a restraint function on the empty graph $G = \overline{K_n}$. Then for all $x \geq M_{G,r}$,*

$$\pi_r(G, x) = \prod_{v \in V(G)} (x - |r(v)|) = \sum_{i=0}^{n} (-1)^{n-i} S_i(A) x^{n-i}$$

*where $A = [|r(v)| : v \in V(G)]$.*