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On the partial order competition dimensions of chordal graphs

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ABSTRACT

Choi et al. (2016) introduced the notion of the partial order competition dimension of a graph. It was shown that complete graphs, interval graphs, and trees, which are chordal graphs, have partial order competition dimensions at most three.

In this paper, we study the partial order competition dimensions of chordal graphs. We show that chordal graphs have partial order competition dimensions at most three if the graphs are diamond-free. Moreover, we also show the existence of chordal graphs containing diamonds whose partial order competition dimensions are greater than three.

1. Introduction

The *competition graph* $C(D)$ of a digraph D is an undirected graph which has the same vertex set as D and which has an edge xy between two distinct vertices x and y if and only if for some vertex $z \in V$, the arcs (x, z) and (y, z) are in D.

Let d be a positive integer. For $x = (x_1, x_2, \dots, x_d), y = (y_1, y_2, \dots, y_d) \in \mathbb{R}^d$, we write $x \prec y$ if $x_i < y_i$ for each $i = 1, \dots, d$. For a finite subset S of $\mathbb{R}^d$, let D_S be the digraph defined by $V(D_S) = S$ and $A(D_S) = \{(x, v) \mid v, x \in S, v \prec x\}$. A digraph D is called a *d-partial order* if there exists a finite subset S of $\mathbb{R}^d$ such that D is isomorphic to the digraph D_S. A 2-partial order is also called a *doubly partial order*. Cho and Kim [2] studied the competition graphs of doubly partial orders and showed that interval graphs are exactly the graphs having partial order competition dimensions at most two. Especially, Wu and Lu [10] answered an open problem posed by Cho and Kim [2] of characterizing competition graphs of d-partial orders for $d \leq 2$. Several variants of competition graphs of doubly partial orders also have been studied (see [4–9]).

Choi et al. [3] introduced the notion of the partial order competition dimension of a graph, which had been also implicitly introduced by Wu and Lu [10] (refer to Remark 3.4 in [3] for further details).

Definition. For a graph G, the *partial order competition dimension* of G, denoted by $\dim_{\text{poc}}(G)$, is the smallest nonnegative integer d such that G together with k isolated vertices is the competition graph of a d-partial order D for some nonnegative integer k, i.e.,

$$\dim_{\text{poc}}(G) := \min\{d \in \mathbb{Z}_{\geq 0} \mid \exists k \in \mathbb{Z}_{\geq 0}, \exists S \subseteq \mathbb{R}^d \text{ s.t. } G \cup I_k = C(D_S)\},$$

where $\mathbb{Z}_{\geq 0}$ is the set of nonnegative integers and I_k is a set of k isolated vertices.

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Choi et al. [3] studied graphs having small partial order competition dimensions, and gave characterizations of graphs with partial order competition dimension 0, 1, or 2 as follows.

Proposition 1.1. *Let G be a graph. Then,* $\dim_{\mathrm{poc}}(G) = 0$ *if and only if* $G = K_1$.

Proposition 1.2. *Let G be a graph. Then,* $\dim_{\mathrm{poc}}(G) = 1$ *if and only if* $G = K_{t+1}$ *or* $G = K_t \cup K_1$ *for some positive integer* t.

Proposition 1.3. *Let G be a graph. Then,* $\dim_{\mathrm{poc}}(G) = 2$ *if and only if G is an interval graph which is neither K_s nor $K_t \cup K_1$ for any positive integers s and t.*

Choi et al. [3] also gave some families of graphs with partial order competition dimension three.

Proposition 1.4. *If G is a cycle of length at least four, then* $\dim_{\mathrm{poc}}(G) = 3$.

A *caterpillar* is a tree the removal of whose pendant vertices results in a path.

Theorem 1.5. *Let T be a tree. Then* $\dim_{\mathrm{poc}}(T) \leq 3$, *and the equality holds if and only if T is not a caterpillar.*

In this paper, we study the partial order competition dimensions of chordal graphs. We thought that most likely candidates for the family of graphs having partial order competition dimension at most three are chordal graphs since both trees and interval graphs, which are chordal graphs, have partial order competition dimensions at most three. In fact, we show that chordal graphs have partial order competition dimensions at most three if the graphs are diamond-free. However, contrary to our presumption, we could show the existence of chordal graphs with partial order competition dimensions greater than three.

2. Preliminaries

We say that two sets in $\mathbb{R}^d$ are *homothetic* if they are related by a geometric contraction or expansion. Choi et al. [3] gave a characterization of the competition graphs of d-partial orders. We state it in the case where $d = 3$.

Theorem 2.1 ([3]). *A graph G is the competition graph of a 3-partial order if and only if there exists a family $\mathcal{F}$ of homothetic open equilateral triangles contained in the plane $\{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0\}$ and there exists a one-to-one correspondence $A : V(G) \to \mathcal{F}$ such that*

($\star$) *two vertices v and w are adjacent in G if and only if two elements $A(v)$ and $A(w)$ have the intersection containing the closure $\triangle(x)$ of an element $A(x)$ in $\mathcal{F}$.*

Choi et al. [3] also gave a sufficient condition for a graph being the competition graph of a d-partial order. We state their result in the case where $d = 3$.

Theorem 2.2 ([3]). *If G is the intersection graph of a finite family of homothetic closed equilateral triangles, then G together with sufficiently many new isolated vertices is the competition graph of a 3-partial order.*

By the definition of the partial order competition dimension of a graph, we have the following:

Corollary 2.3. *If G is the intersection graph of a finite family of homothetic closed equilateral triangles, then* $\dim_{\mathrm{poc}}(G) \leq 3$.

Note that the converse of Corollary 2.3 is not true by an example given by Choi et al. [3] (see Fig. 1). In this context, one can guess that it is not so easy to show that a graph has partial order competition dimension greater than three.

The correspondence A in Theorem 2.1 can be precisely described as follows: Let $\mathcal{H} := \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1+x_2+x_3 = 0\}$ and $\mathcal{H}_+ := \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1+x_2+x_3 > 0\}$. For a point $v = (v_1, v_2, v_3) \in \mathcal{H}_+$, let $p_1^{(v)}$, $p_2^{(v)}$, and $p_3^{(v)}$ be points in $\mathbb{R}^3$ defined by $p_1^{(v)} := (-v_2 - v_3, v_2, v_3)$, $p_2^{(v)} := (v_1, -v_1 - v_3, v_3)$, and $p_3^{(v)} := (v_1, v_2, -v_1 - v_2)$, and let $\triangle(v)$ be the convex hull of the points $p_1^{(v)}$, $p_2^{(v)}$, and $p_3^{(v)}$, i.e., $\triangle(v) := \mathrm{Conv}(p_1^{(v)}, p_2^{(v)}, p_3^{(v)}) = \left\{\sum_{i=1}^{3} \lambda_i p_i^{(v)} \mid \sum_{i=1}^{3} \lambda_i = 1, \lambda_i \geq 0\ (i = 1, 2, 3)\right\}$. Then it is easy to check that $\triangle(v)$ is an closed equilateral triangle which is contained in the plane $\mathcal{H}$. Let $A(v)$ be the relative interior of the closed triangle $\triangle(v)$, i.e., $A(v) := \mathrm{rel.int}(\triangle(v)) = \left\{\sum_{i=1}^{3} \lambda_i p_i^{(v)} \mid \sum_{i=1}^{3} \lambda_i = 1, \lambda_i > 0\ (i = 1, 2, 3)\right\}$. Then $A(v)$ and $A(w)$ are homothetic for any $v, w \in \mathcal{H}_+$.

For $v \in \mathcal{H}_+$ and $(i, j) \in \{(1, 2), (2, 3), (1, 3)\}$, let $l_{ij}^{(v)}$ denote the line through the two points $p_i^{(v)}$ and $p_j^{(v)}$, i.e., $l_{ij}^{(v)} := \{x \in \mathbb{R}^3 \mid x = \alpha p_i^{(v)} + (1 - \alpha) p_j^{(v)}, \alpha \in \mathbb{R}\}$, and let $R_{ij}(v)$ denote the following region:

$$R_{ij}(v) := \{x \in \mathbb{R}^3 \mid x = (1 - \alpha - \beta) p_k^{(v)} + \alpha p_i^{(v)} + \beta p_j^{(v)}, 0 \leq \alpha \in \mathbb{R}, 0 \leq \beta \in \mathbb{R}, \alpha + \beta \geq 1\},$$

where k is the element in $\{1, 2, 3\} \setminus \{i, j\}$; for $k \in \{1, 2, 3\}$, let $R_k(v)$ denote the following region:

$$R_k(v) := \{x \in \mathbb{R}^3 \mid x = (1 + \alpha + \beta) p_k^{(v)} - \alpha p_i^{(v)} - \beta p_j^{(v)}, 0 \leq \alpha \in \mathbb{R}, 0 \leq \beta \in \mathbb{R}\},$$

where i and j are elements such that $\{i, j, k\} = \{1, 2, 3\}$. (See Fig. 2 for an illustration.)

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