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Secure total domination in graphs: Bounds and complexity

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ABSTRACT

A total dominating set of an undirected graph $G = (V, E)$ is a set $S \subseteq V$ of vertices such that each vertex of G is adjacent to at least one vertex of S. A secure total dominating set of G is a set $D \subseteq V$ of vertices such that D is a total dominating set of G and each vertex $v \in V \setminus D$ is adjacent to at least one vertex $u \in D$ with the property that the set $(D \setminus \{u\}) \cup \{v\}$ is a total dominating set of G. The secure total domination number $\gamma_{st}(G)$ of G is the minimum cardinality of a secure total dominating set of G. We establish bounds on the secure total domination number. In particular, we show that every graph G with no isolated vertices satisfies $\gamma_{st}(G) \leq 2\alpha(G)$, where $\alpha(G)$ is the independence number of G. Further, we study the problem of finding the secure total domination number. We show that the decision version of the problem is NP-complete for chordal bipartite graphs, planar bipartite graphs with arbitrary large girth and maximum degree 3, split graphs and graphs of separability at most 2. Finally, we show that the optimisation version of the problem can be approximated in polynomial time within a factor of $c \ln |V|$ for some constant $c > 1$ and cannot be approximated in polynomial time within a factor of $c' \ln |V|$ for some constant $c' < 1$, unless $P = NP$.

1. Introduction

We consider only finite undirected graphs $G = (V, E)$ with vertex set $V = V(G)$ and edge set $E = E(G)$ without loops or multiple edges and use standard graph-theoretic terminology and notations (see, for example, [9]). We say that a vertex $v \in V$ is dominated by a vertex $u \in V$ if the vertex u is adjacent to the vertex v. A vertex subset $D \subseteq V$ of G is a *dominating set* of G if each vertex $v \in V \setminus D$ is dominated by at least one vertex of D, and a *total dominating set* of G if each vertex $v \in V$ is dominated by at least one vertex of D. The minimum cardinality of a (total) dominating set of G is denoted by $\gamma(G)$ (respectively, $\gamma_t(G)$). The domination number $\gamma(G)$ and the total domination number $\gamma_t(G)$ have been extensively studied [10,11,13,12].

Fairly recently, secure versions of (total) dominating sets of a graph have been introduced [1,6]. Let $S \subseteq V$ be a dominating set of a graph $G = (V, E)$ and $v \in V \setminus S$, $u \in S$. We say that the vertex u *defends* the vertex v if the vertices u and v are adjacent and $(S \setminus \{u\}) \cup \{v\}$ is a dominating set of G. A vertex subset $D \subseteq V$ of G is a *secure dominating set* of G if D is a dominating set of G and each vertex $v \in V \setminus D$ is defended by at least one vertex of D. The minimum cardinality of a secure dominating set of G is called the *secure domination number* of G and is denoted by $\gamma_s(G)$. Let $S \subseteq V$ be a total dominating set of $G = (V, E)$ and $v \in V \setminus S$, $u \in S$. We say that the vertex u *totally defends* the vertex v (and also that the vertex u is a *total defender* of the vertex v) if the vertices u and v are adjacent and $(S \setminus \{u\}) \cup \{v\}$ is a total dominating set of G. A vertex subset

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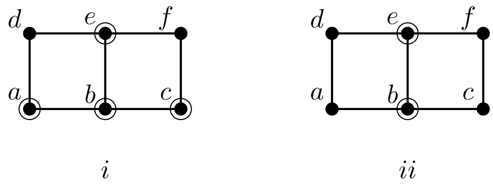

Fig. 1. (i) The set $\{a, b, c, e\}$ is secure total dominating; (ii) the set $\{b, e\}$ is total dominating but not secure total dominating.

$D \subseteq V$ of G is a *secure total dominating set* of G if D is a total dominating set of G and each vertex $v \in V \setminus D$ is totally defended by at least one vertex of D. In other words, a total dominating set D of G is secure if for each vertex $v \in V \setminus D$ there is at least one vertex $u \in D$ that is a total defender of v. The minimum cardinality of a secure total dominating set of G is called the *secure total domination number* of G and is denoted by $\gamma_{st}(G)$. The secure total domination number is defined solely for graphs with no isolated vertices [15].

To illustrate the definition of a secure total dominating set, we consider the graph shown in Fig. 1(i). The set $D_1 = \{a, b, c, e\}$ (see Fig. 1(i)) is a secure total dominating set of the graph (D_1 is a total dominating set; the vertex a is a total defender of the vertex d and the vertex c is a total defender of the vertex f). At the same time, the set $D_2 = \{b, e\}$ (see Fig. 1(ii)) is total dominating, but not secure total dominating (for example, the vertex a does not have a total defender since there is exactly one vertex of D_2, namely the vertex b, that is adjacent to the vertex a, but the set $(D_2 \setminus \{b\}) \cup \{a\} = \{a, e\}$ is not even a dominating set of the considered graph).

We study the secure total domination number of a graph and the following problem related to this graph parameter:

Secure Total Dominating Set

Instance: A graph $G = (V, E)$ without isolated vertices and a positive integer k.

Question: Is there a secure total dominating set D of G such that $|D| \leq k$? Equivalently, is $\gamma_{st}(G) \leq k$?

The optimisation version of Secure Total Dominating Set asks for a minimum cardinality secure total dominating set of a given graph without isolated vertices.

The secure total domination number of a graph was introduced by Benecke et al. in [1]. This graph parameter is studied in papers [1,3,8,15]. Klostermeyer et al. in [15] compare the secure (total) domination numbers with other graph parameters and characterise graphs with equal total and secure total domination numbers. Castillano et al. [3] and Go et al. [8] study secure (total) domination numbers in graphs that are obtained by means of the following graph operations: the join, the corona and the composition of graphs. To the best of our knowledge, the computational complexity of the Secure Total Dominating Set problem has not been studied.

The paper is organised as follows. In Section 2, we first characterise secure total dominating sets in $(P_5, bull)$-free graphs and then we disprove a conjecture on the secure total domination number of the composition $G[K_n]$ given in [3] and answer an open question concerning the sharpness of an upper bound on the secure total domination number posed by Klostermeyer et al. in [15]. In Section 3, we show that the Secure Total Dominating Set problem is NP-complete for chordal bipartite graphs, planar bipartite graphs with girth at least p and maximum degree 3, split graphs and graphs of separability at most 2. In Section 4, we show that the optimisation version of the studied problem can be approximated in polynomial time within a factor of $c \ln |V|$ for some constant $c > 1$ and cannot be approximated in polynomial time within a factor of $c' \ln |V|$ for some constant $c' < 1$, unless $P = NP$.

2. Secure total dominating sets in $(P_5, bull)$-free graphs and the composition $G[K_n]$, bounds

A *leaf* of a graph G is a vertex of degree 1, a *support vertex* of G is a vertex that is adjacent to a leaf. Let D be a secure total dominating set of G. As, by definition, D is a total dominating set of G, each support vertex of G is contained in D. Let us observe that each leaf of G is contained in D as well. If some leaf $\ell \notin D$, for this leaf ℓ there is no total defender. It implies the following simple and important observation.

Observation 1. *Let $G = (V, E)$ be a graph without isolated vertices. Every secure total dominating set of G contains each support vertex of G and each leaf of G.*

The graph *bull* is the graph with the vertex set $\{a, b, c, d, e\}$ and the edge set $\{\{a, b\}, \{b, c\}, \{a, c\}, \{b, e\}, \{c, d\}\}$. Now, we consider $(P_5, bull)$-free graphs, i.e., graphs without induced subgraphs isomorphic to simple 5-vertex path P_5 or the graph *bull*. We give a characterisation of secure total dominating sets in $(P_5, bull)$-free graphs without isolated vertices. As usual, $N_G(v)$ denotes the set of all vertices of G adjacent to a vertex $v \in V(G)$ and $N_G[v] = N_G(v) \cup \{v\}$. The notation $u \sim v$ ($u \nsim v$, respectively) means that vertices u and v are adjacent (non-adjacent, respectively).

Lemma 1. *Let $G = (V, E)$ be a $(P_5, bull)$-free graph without isolated vertices. For a vertex subset $D \subseteq V$ of G the following statements are equivalent:*

(1) *D is a secure total dominating set of G;*
(2) *$|N_G[v] \cap D| \geq 2$ for each vertex v of G.*