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Relation between the skew energy of an oriented graph and its matching number

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ABSTRACT

Let G^σ be an oriented graph with skew adjacency matrix $S(G^\sigma)$. The skew energy $\mathcal{E}_S(G^\sigma)$ of G^σ is the sum of the norms of all eigenvalues of $S(G^\sigma)$ and the skew rank $sr(G^\sigma)$ of G^σ is the rank of $S(G^\sigma)$. In this paper, it is proved that $\mathcal{E}_S(G^\sigma) \geq 2\mu(G)$ for an arbitrary connected oriented graph G^σ of order n, where $\mu(G)$ is the matching number of G, and the equality holds if and only if G is a complete bipartite graph $K_{\frac{n}{2},\frac{n}{2}}$ with partition (X, Y) of equal size and σ is switching-equivalent to the elementary orientation of G which assigns all edges the same direction from vertices of X to vertices of Y. As an application, we prove that $\mathcal{E}_S(G^\sigma) \geq sr(G^\sigma)$ for an oriented graph G^σ and the equality holds if and only if G is the disjoint union of some copies of K_2 and some isolated vertices.

1. Introduction

Analogous to the definition of the energy of a simple undirected graph, Adiga et al. [1] defined the skew energy $\mathcal{E}_S(G^\sigma)$ of an oriented graph G^σ, with adjacency matrix $S(G^\sigma)$, to be the sum of the norms of all eigenvalues of $S(G^\sigma)$. Since $S(G^\sigma)$ is skew symmetric, an eigenvalue of $S(G^\sigma)$ is either a pure imaginary number or 0, thus the norm of an eigenvalue of $S(G^\sigma)$ is a singular value of $S(G^\sigma)$ and $\mathcal{E}_S(G^\sigma)$ also equals the sum of all singular values of $S(G^\sigma)$.

Adiga et al. [1] established a low bound and an upper bound for the skew energy of an oriented graph G^σ in terms of the order and size of G^σ as well as the maximum degree of its underlying graph.

Proposition 1.1 (*Theorem 2.5, [1]*). *Let G^σ be an oriented graph of G with n vertices, m arcs and maximum degree Δ. Then*

$$\sqrt{2m + n(n-1)p^{\frac{2}{n}}} \leq \mathcal{E}_S(G^\sigma) \leq \sqrt{2mn} \leq n\sqrt{\Delta},$$

where $p = |\det(S(G^\sigma))|$.

The upper bound $n\sqrt{\Delta}$ of $\mathcal{E}_S(G^\sigma)$ is called the optimum skew energy. It was proved that if an oriented graph G^σ has the optimum skew energy, then G is a Δ-regular graph. Hence a natural question was posed in [1]: *Which k-regular graphs G on n vertices have an orientation σ such that $\mathcal{E}_S(G^\sigma) = n\sqrt{k}$?* For a small k, some results have been obtained. If $k \leq 2$, the authors in [1] characterized the k-regular graphs which has an orientation such that $\mathcal{E}_S(G^\sigma) = n\sqrt{k}$. Gong and Xu [10] characterized all 3-regular oriented graphs with optimum skew energy. Chen et al. [5] and Gong et al. [11] independently determined the underlying graphs of all 4-regular oriented graphs with optimum skew energy and gave orientations of these underlying

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graphs such that the skew energies of the resultant oriented graphs indeed attain optimum. Chen et al. [6] obtained some lower bounds of the skew energy of G^σ, which improves the known lower bound obtained by Adiga et al. [1]. For more recent results on skew adjacency matrix or skew energy of oriented graphs we refer the reader to [3,4,9,13–18].

In this paper, we study the relation between the skew energy $\mathcal{E}_S(G^\sigma)$ of an oriented graph G^σ and its matching number. In Section 2, we prove that $\mathcal{E}_S(G^\sigma) \geq 2\mu(G)$ for a connected oriented graph G^σ with matching number $\mu(G)$. In Section 3, we characterize the oriented graphs G^σ with equality $\mathcal{E}_S(G^\sigma) = 2\mu(G)$, proving that the equality holds if and only if G is a complete bipartite graph with equal bipartite (X, Y) and σ is switching-equivalent to the elementary orientation of G which assigns all edges the same direction from vertices of X to those of Y. As an application, we prove that $\mathcal{E}_S(G^\sigma) \geq sr(G^\sigma)$ for an oriented graph G^σ and the equality holds if and only if G is the disjoint union of some copies of K_2 and some isolated vertices.

2. A lower bound of the skew energy of an oriented graph in terms of matching number

Let G be a simple graph of order n with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. Based on G, we define an oriented graph G^σ obtained from G by assigning each edge of G a direction, where G is called the underlying graph of G^σ. The skew adjacency matrix associated to G^σ, written as $S(G^\sigma)$, is defined to be an $n \times n$ matrix (s_{ij}) such that $s_{ij} = 1$ if there is an arc from v_i to v_j, $s_{ij} = -1$ if there is an arc from v_j to v_i and $s_{ij} = 0$ otherwise. The skew rank of G^σ, denoted by $sr(G^\sigma)$, is defined to be the rank of $S(G^\sigma)$, which is even since $S(G^\sigma)$ is skew symmetric. Let $C_k^\sigma = u_1u_2\cdots u_ku_1$ be an oriented cycle. The sign of C_k^σ, denoted by $sgn(C_k^\sigma)$, is defined to be the sign of $(\Pi_{i=1}^{k-1} s_{u_iu_{i+1}})s_{u_ku_1}$. An even oriented cycle C_{2m}^σ is called evenly-oriented (resp., oddly-oriented) if its sign is positive (resp., negative). If every even cycle in G^σ is evenly-oriented, then G^σ is called evenly-oriented. An induced subgraph H^σ of G^σ is an oriented graph such that H is an induced subgraph of G and each arc of H^σ has the same orientation as that in G^σ. For $U \subseteq V(G^\sigma)$, $G^\sigma - U$ is the subgraph obtained from G^σ by deleting all vertices in U and all incident arcs. A vertex of a graph G^σ is called pendant if it is only adjacent to one vertex. A set M of edges in G is a matching if every vertex of G is incident with at most one edge in M. It is a perfect matching if every vertex of G is incident with exactly one edge in M. We denote by $\mu(G)$ the matching number of G (i.e. the number of edges of a maximum matching in G). We respectively use $P_n, C_n, K_n, K_{p,n-p}$ to denote a path, a cycle, a complete graph and a complete bipartite graph on n vertices.

Let $s_j(C)$ denote a singular value of a complex matrix C. Day and So [8] obtained an inequality about a partitioned matrix C as follows.

Lemma 2.1 (*Theorem 2.2, [8]*). *For a partitioned matrix* $C = \begin{bmatrix} A & X \\ Y & B \end{bmatrix}$, *where both A and B are square matrices, we have* $\sum_j s_j(A) + \sum_j s_j(B) \leq \sum_j s_j(C)$. *Equality holds if and only if there exist unitary matrices U and V such that* $\begin{bmatrix} UA & UX \\ VY & VB \end{bmatrix}$ *is positive semi-definite.*

Lemma 2.2 (*Corollary 2.4, [8]*). *For a partitioned matrix* $C = \begin{bmatrix} A & X \\ Y & B \end{bmatrix}$, *where both A and B are square matrices, we have* $\sum_j s_j(A) \leq \sum_j s_j(C)$. *Equality holds if and only if X, Y and B are all zero matrices.*

Applying Lemma 2.2 to an induced subgraph H^σ of an oriented graph G^σ, we obtain a similar result as Theorem 3.1 of [8].

Lemma 2.3. *Let H^σ be an induced subgraph of an oriented graph G^σ. Then $\mathcal{E}_S(H^\sigma) \leq \mathcal{E}_S(G^\sigma)$ and equality holds if and only if $E(H) = E(G)$.*

We write $G^\sigma - H^\sigma$ for the oriented graph obtained from G^σ by deleting all vertices of an induced subgraph H^σ and all arcs incident with H^σ. This is also called the complement of H^σ in G^σ. Moreover, when no arcs of G^σ join H^σ and its complement $G^\sigma - H^\sigma$, we write $G^\sigma = H^\sigma \oplus (G^\sigma - H^\sigma)$. If E is a set of arcs of G^σ such that $G^\sigma - E$, the subgraph of G^σ obtained from G^σ by deleting all arcs in E, is the union of two complementary induced subgraphs, then E is called a cut set of G^σ. Theorem 3.4 of [8] proved that the energy of a spanned subgraph of a simple graph G does not exceed that of G when a cut set of G is deleted. A similar result also holds for oriented graphs.

Lemma 2.4. *If E is a cut set of an oriented graph G^σ then $\mathcal{E}_S(G^\sigma - E) \leq \mathcal{E}_S(G^\sigma)$.*

Proof. Since E is a cut set of G^σ, $G^\sigma - E = H^\sigma \oplus K^\sigma$, where H^σ and K^σ are two complementary induced subgraphs of G^σ. Applying Lemma 2.1 to $S(G^\sigma) = \begin{bmatrix} S(H^\sigma) & * \\ * & S(K^\sigma) \end{bmatrix}$, we obtain the desired conclusion. □

Theorem 3.6 of [8] proved that $\mathcal{E}(G - E) < \mathcal{E}(G)$ if E is a cut set of a graph G which forms a star, where $\mathcal{E}(G)$ denotes the energy of G (see [8] for its definition). This result can also be translated to oriented graphs. The proof of the following lemma is analogous to that of Theorem 3.6 of [8], thus omitted.

Lemma 2.5. *If E is a cut set of an oriented graph G^σ such that the arcs of E form an oriented star, then $\mathcal{E}_S(G^\sigma - E) < \mathcal{E}_S(G^\sigma)$.*

Now, we apply Lemma 2.4 to obtain a lower bound of the skew energy of an oriented graph in terms of matching number.