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On b-coloring of powers of hypercubes

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ABSTRACT

A b-coloring of a graph G with k colors is a proper coloring of G using k colors in which each color class contains a color dominating vertex, that is, a vertex which has a neighbor in each of the other color classes. The largest positive integer k for which G has a b-coloring using k colors is the b-chromatic number $b(G)$ of G. In this paper, we have obtained bounds for the b-chromatic number of powers of Q_n, namely Q_n^p, for $n \geq 5$ and $\lfloor \frac{n}{2} \rfloor < p < n-1$. Also we have found the exact value of the b-chromatic number of Q_n^p for $n \geq 3$, and $p = \lfloor \frac{n}{2} \rfloor$ and $p = n-1$. In addition, we have determined the clique number of Q_n^p for $n \geq 3$ and the chromatic number of Q_n^p for $n \geq 2$ and $\left\lceil \frac{2(n-1)}{3} \right\rceil \leq p \leq n-1$.

1. Introduction

All graphs considered in this paper are simple, finite and undirected. A b-coloring of a graph is a proper coloring of the vertices of G such that each color class contains a color dominating vertex (c.d.v.), that is, a vertex which is adjacent to at least one vertex of every other color class. The largest positive integer k for which G has a b-coloring using k colors is the b-chromatic number $b(G)$ of G. From the definition of $\chi(G)$, we observe that each color class of a χ-coloring contains a c.d.v. Thus $\omega(G) \leq \chi(G) \leq b(G)$, where $\omega(G)$ is the size of a maximum clique of G. The concept of b-coloring was introduced by Irving and Manlove [9] in analogy to the achromatic number of a graph G. They have shown that the determination of $b(G)$ is NP-hard for general graphs, but polynomial for trees. Some of the references in b-coloring are [1,6,5,13].

Suppose that the vertices of a graph G are ordered as $v_1, v_2, \ldots, v_n$ such that $d(v_1) \geq d(v_2) \geq \cdots \geq d(v_n)$. Then the m-degree, $m(G)$, of G is defined by $m(G) = \max\{i : d(v_i) \geq i-1,\ 1 \leq i \leq n\}$. For any graph G, $b(G) \leq m(G) \leq \Delta(G)+1$ where $\Delta(G)$ is the maximum degree of G. Also for any regular graph G, $m(G) = \Delta(G)+1$.

Let us recall the definition of strongly regular graphs and b-spectrum of a graph.

A graph G is strongly regular if there are parameters (n, k, λ, μ) such that G has order n, regularity k, every pair of adjacent vertices have λ common neighbors, and every pair of non-adjacent vertices have μ common neighbors.

Graphs which have a b-coloring using k colors, for every k such that $\chi(G) \leq k \leq b(G)$ are known as b-continuous graphs. There are graphs which are not b-continuous. For instance, consider $G = K_{n,n} - 1F$ (complete bipartite graph on $2n$ vertices except a perfect matching), $n \geq 4$. One can observe that G has a b-coloring using 2 colors and n colors but none using k colors where $3 \leq k \leq n-1$. The b-spectrum $S_b(G)$ of a graph G is the set of positive integers k, for which G has a b-coloring using k colors. Clearly, $\{\chi(G), b(G)\} \subseteq S_b(G) \subseteq \{\chi(G), \chi(G)+1, \ldots, b(G)\}$. G is b-continuous if and only if $S_b(G) = \{\chi(G), \chi(G)+1, \ldots, b(G)\}$.

The n-hypercube graph denoted by Q_n is the graph whose vertices are the 2^n symbols $a_1a_2\ldots a_n$ where $a_i = 0$ or 1 and two vertices are adjacent if the symbols differ in exactly one coordinate. If $v \in V(Q_n)$, then $\bar{v}$ denotes the complement of v

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got by replacing 0 by 1 and 1 by 0 in v. In case of no ambiguity we write hypercube instead of n-hypercube. For any integer $p \geq 1$, the pth power of a graph G denoted by G^p is a graph obtained from G by adding an edge between every pair of vertices at a distance of p or less. It is easy to see that $G^1 = G$. Powers of several graph classes have been investigated in the past. See for instance [3,4,15,16]. The b-chromatic number of powers of paths, cycles and complete caterpillars have been studied in [6,5,14]. The problem of finding exact value of the chromatic number for Q_n^p seems to be a challenging one. This has lead to finding bounds for the chromatic number of Q_n^p. This can be seen in [11,15,16]. The coloring problem on hypercubes and its powers has been extensively studied and has a vast number of applications to multi computer networks and distributed computation [2,8].

In this paper, we have obtained bounds for the b-chromatic number of Q_n^p for $n \geq 5$ and $\left\lfloor \frac{n}{2} \right\rfloor < p < n-1$. Also we have found the exact value of $b(Q_n^p)$ all $n \geq 3$, and $p = \left\lfloor \frac{n}{2} \right\rfloor$ and $p = n-1$. In addition, by using Erdös–Ko–Rado theorem on intersecting families [7] and a result of D.J. Kleitman (see [12]), we have determined the clique number of Q_n^p for $n \geq 3$. Finally we have shown that the coloring technique used for finding the b-chromatic number of $Q_n^{\left\lfloor \frac{n}{2} \right\rfloor}$ helps us in showing that $\chi(Q_n^p) = 2^{n-1}$ for $n \geq 2$ and $\left\lceil \frac{2(n-1)}{3} \right\rceil \leq p \leq n-1$.

2. Bounds for the b-chromatic number of powers of hypercubes

The following are some observations that can be made of Q_n^p.

Observation 2.1. *The graph Q_n^p is*

(i) $\sum_{i=1}^{p} \binom{n}{i}$ *regular and vertex-transitive.*

(ii) *The diameter of Q_n^p is $\left\lceil \frac{n}{p} \right\rceil$.*

(iii) *For $p \geq n$, $b(Q_n^p) = 2^n$.*

Let us start by finding the b-chromatic number of Q_n^{n-1}.

Theorem 2.2. *The b-chromatic number of Q_n^{n-1} is 2^{n-1} for all $n \geq 2$.*

Proof. Let us consider the graph Q_n^{n-1}. Since $\bar{v}$ is the only vertex at distance n from v in Q_n, each vertex v is non-adjacent only to its complement $\bar{v}$. Hence the graph Q_n^{n-1} is isomorphic to the complement of a perfect matching on 2^n vertices. The graph Q_n^{n-1} has a clique of size 2^{n-1}, which implies that $b(Q_n^{n-1}) \geq 2^{n-1}$. In any b-coloring of Q_n^{n-1} the non-adjacent vertices v and $\bar{v}$ must receive the same color since all the other vertices are common neighbors of both v and $\bar{v}$. Also there are 2^{n-1} such pairs. Hence $b(Q_n^{n-1}) = 2^{n-1}$. □

Fact 2.3. (i) $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$, $1 \leq k \leq n$.

(ii) *If n is even then* $2\sum_{i=0}^{\frac{n}{2}-1} \binom{n-1}{i} = 2^{n-1} = \sum_{i=0}^{\frac{n}{2}-1} \binom{n}{i} + \binom{n}{\frac{n}{2}}/2$.

Next, we shall find the b-chromatic number of $Q_n^{\left\lfloor \frac{n}{2} \right\rfloor}$. For doing this, we first prove the following lemma on the number of common neighbors of any two adjacent vertices in powers of hypercubes.

Lemma 2.4. *For $n \geq 2$ and $1 \leq p \leq n$, the number of common neighbors of any two adjacent vertices in Q_n^p is at most $2\sum_{i=1}^{p-1} \binom{n-1}{i}$. Equality holds if and only if $p = 2, n-1, n$.*

Proof. Let $\lambda(x, y)$ denote the number of common neighbors of any two adjacent vertices x and y in G, where $G = Q_n^p$. For $p = 1$, $\lambda(x, y) = 0$ for all $xy \in E(Q_n)$. For $p = n-1$, as mentioned earlier the graph Q_n^{n-1} is isomorphic to the complement of a perfect matching on 2^n vertices. Clearly for any two adjacent vertices $x, y \in V(Q_n^{n-1})$, $\lambda(x, y) = 2^n - 4 = 2\sum_{i=1}^{n-2} \binom{n-1}{i}$. Finally when $p = n$, Q_n^p is isomorphic to K_{2^n} and hence equality follows immediately.

Now let us consider p such that $2 \leq p \leq n-2$. Since Q_n is distance-transitive, Q_n is also distance-regular and hence for any two vertices v and w, the number of vertices at distance j from v and at distance k from w depends only upon j, k, and $i = d(v, w)$. Thus in Q_n, for any two positive integers l and p, the number of vertices which are at a distance of at most p from any two vertices whose distance is l will be same. Therefore it will suffice to find the number of common neighbors of $00\ldots0$ with its adjacent vertices in Q_n^p. Let us start by considering $u = 00\ldots0$. Also let V_i, $0 \leq i \leq n$ denote the set of vertices which are at a distance of i from u in Q_n. Let $v = b_1b_2\cdots b_n \in V_l$, $1 \leq l \leq p$ and let $I = \{k \mid 1 \leq k \leq n \text{ and } b_k = 1\}$ and $J = \{k \mid 1 \leq k \leq n \text{ and } b_k = 0\}$. Clearly $|I| = l$ and $|J| = n - l$. Now let Z denote the set of vertices that differ from v at i places in I and j places in J. Here it is not difficult to observe the following.

(A) $|Z| = \binom{l}{i}\binom{n-l}{j}$

(B) $Z \subseteq V_{l-i+j}$

(C) the vertices in Z are the only vertices which are at a distance of $i+j$ from v belonging to V_{l-i+j} in Q_n.