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# A complete characterization of jump inequalities for the hop-constrained shortest path problem

Wolfgang F. Riedl

*Zentrum Mathematik, Technische Universität München, Boltzmannstr. 3, D-85748, Garching bei München, Germany*

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ABSTRACT

Considering an integer $k$ and a directed, weighted graph with two distinct nodes $s$ and $t$, the *Hop-Constrained Shortest Path Problem* looks for a shortest $(s, t)$-path using at most $k$ arcs. In this paper, we study the polytope of the convex hull of incidence vectors of $(s, t)$-paths using at most $k$ arcs. We present valid inequalities and adaptations of known concepts such as cloning and a unique representation of facets. The main focus will be on one particular family of valid inequalities, the family of *Jump Inequalities*. Thereby, the main contribution will be a characterization of the members of the family inducing facets. Furthermore, all possibilities to lift the remaining inequalities will be defined. The analysis of Jump Inequalities will be concluded by further results concerning the equivalence of inequalities as well as their Chvátal rank.



## 1. Introduction

### *1.1. The hop-constrained shortest path problem*

For a given weighted, directed graph $G = (V, A, c)$ with cost function $c : A \to \mathbb{R}$, two distinct nodes $s \neq t \in V$ and an integer $k \geq 3$, the *Hop-Constrained Shortest Path Problem* (HSPP) asks for a minimal $(s, t)$-path (with respect to $c$) that uses at most $k$ arcs. Thereby, given two distinct nodes $s \neq t \in V$, an $(s, t)$-path is defined as a collection of nodes $P = (v_1, \dots, v_l)$ such that $v_1 = s$, $v_l = t$, $(v_i, v_{i+1}) \in A\ \forall i \in \{1, \dots, l-1\}$ and $v_i \neq v_j\ \forall i \neq j \in \{1, \dots, l\}$.

Note that paths are assumed to be elementary, thus a path may not visit nodes more than once. Since any path on $|V|$ nodes has at most $|V| - 1$ arcs, we will w.l.o.g. assume that

$$k \leq |V| - 1.$$

The Hop-Constrained Shortest Path Problem appears as a subproblem in the context of Column Generation or Lagrangian Relaxation, mostly in telecommunication network design problems. In this problem setting, hop-constraints assert the quality of the networks to be constructed by bounding the probability of losing one of the data packages to be transferred (see for instance [1,14,15,9]). If $c((u, v)) \geq 0\ \forall (u, v) \in A$, the problem is easily solvable in polynomial time (cf. [12, ND30] or [20] for an early algorithm). However, in the scenario of Column Generation or Lagrangian Relaxation, this assumption is not true in general.

If $c((u, v)) \in \mathbb{R}\ \forall (u, v) \in A$, negative arc weights can be interpreted as an incentive for the subproblem to visit distinct nodes needed in the previous iteration of the master problem. In these cases only non-elementary optimal solutions may exist and standard labelling algorithms (cf. [19] for an overview) solving the relaxed problem of finding a non-elementary

*E-mail address:* riedl@ma.tum.de.

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path will return undesired results. Similar problems arise when using Lagrangian Relaxation to relax constraints connecting different paths in the master problem.

These observations coincide with the fact that the problem with arbitrary arc weights is NP-complete, as on a graph with negative arc weight $c$, the Directed Hamiltonian Path Problem with start and end node (see, e.g., [12, GT39]) is equivalent to the Hop-Constrained Shortest Path Problem (as already mentioned by [8] or for a related problem by [24]). One can still modify labelling algorithms to incorporate the requirement of elementary paths, however in practice this leads to an increase in computation times allowing only small instances to be solved. On the other hand, the solutions obtained by the relaxed problem searching non-elementary paths will usually contain cycles and are therefore not desirable, as this weakens the bound obtained by the master problem of the Column Generation applied. This is why a closer look at the geometrical structure of the polytope of hop-constrained paths is important.

The rest of this work is organized as follows. In Section 2 we will outline results concerning the equivalence of facets as well as the cloning of facets. These concepts have originally been developed for the *Travelling Salesman Problem* and will be transferred to the HSPP in this work. Known facets will be summarized in Section 3. In Section 4 further details concerning one family of inequalities – namely Jump Inequalities – will be presented. We will introduce the main contribution of this work, a complete characterization of facet-defining Jump Inequalities as well as all possibilities to lift them in case they do not define facets. The section will be concluded by further characteristics of Lifted Jump Inequalities.

### *1.2. Notation and mathematical model*

In the following, we will consider the HSPP on a directed weighted graph $D_n = (V, A, c)$ where $n := |V|$, $s \in V$ is the start or source node and $t \in V$ is the end or target node. The arc set $A := \left\{(u, v) \in V^2 : u \neq v, u \neq t, v \neq s\right\}$ will contain all possible arcs but the ones entering $s$ or leaving $t$.

The following notation will be used for certain sets of arcs:

- $\delta^+(S) := \{(u, v) \in A : u \in S, v \in V \setminus S\}$ will define the set of outgoing arcs of node set $S \subset V$. For $\delta^+(\{s\})$ we will write $\delta^+(s)$ for short.
- $\delta^-(S) := \{(u, v) \in A : u \in V \setminus S, v \in S\}$ denotes the set of incoming arcs of node set $S \subset V$. For $\delta^-(\{s\})$ we will write $\delta^-(s)$.
- For $S, T \subseteq V$ we will write $(S : T) := \{(u, v) \in A : u \in S, v \in T, u \neq v\}$ for the set of all arcs starting in $S$ and ending in $T$.
- Given a weight $x : A \to \mathbb{R}$ on the arcs (as for example the value of a variable or the given cost function) we use the short notation $x_{uv} := x((u, v))$ for the weight of arc $(u, v) \in A$.
- When considering some set of arcs $A' \subseteq A$ we will denote by $x(A') := \sum_{(u,v)\in A'} x_{uv}$ the accumulated value of weight $x$ on these arcs.

For an integer $n \in \mathbb{N}$, we denote all positive integers less or equal than $n$ by

$$[n] := \{1, \ldots, n\}.$$

Given an inequality $\alpha^\top x \leq \alpha_0$ in $\mathbb{R}^n$ with $\alpha \in \mathbb{R}^n$, $\alpha_0 \in \mathbb{R}$, we denote by

$$F(\alpha, \alpha_0) := \left\{x \in \mathbb{R}^n : \alpha^\top x = \alpha_0\right\}$$

the corresponding hyperplane. For a vector $\alpha \in \mathbb{R}^n$ and a set $T \subset [n]$, we use

$$\alpha_T := (\alpha_i)_{i\in T}$$

to name the subvector defined by index set $T$.

Last, for a given set $S$, a subset $S' \subset S$ and an element $i \in S$ we use the characteristic function

$$\mathbb{1}_{S'}(i) := \begin{cases} 1 & \text{if } i \in S' \\ 0 & \text{if } i \notin S' \end{cases}$$

that takes value 1 if and only if element $i$ is contained in $S'$.

To define the polytope of hop-constrained paths, we will first model the corresponding optimization problem of finding the optimal path with respect to $c$ using at most $k$ arcs. For that purpose, we will use binary variables $x_{uv}$ for all $(u, v) \in A$ with

$$x_{uv} := \begin{cases} 1 & \text{if } (u, v) \text{ is part of the path} \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The vector $x \in \{0, 1\}^{|A|}$ defined in (1) will also be called the *incidence vector* of the corresponding path, the two terms will be used synonymously. Given an inequality $\alpha^\top x \leq \alpha_0$ and a path, we will denote the path as *tight* if for the corresponding