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The connection between polynomial optimization, maximum cliques and Turán densities

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ABSTRACT

In 1965, Motzkin–Straus established the connection between the maximum cliques and the Lagrangian of a graph, the maximum value of a quadratic function determined by a graph in the standard simplex. This connection gave a proof of the Turán's classical result on Turán densities of complete graphs. In 1980's, Sidorenko and Frankl–Füredi further developed this method for hypergraph Turán problems. However, the connection between the Lagrangian and the maximum cliques of a graph cannot be extended to hypergraphs. In 2009, S. Rota Bulò and M. Pelillo defined a homogeneous polynomial function of degree r determined by an r-uniform hypergraph and gave the connection between the minimum value of this polynomial function and the maximum cliques of an r-uniform hypergraph. In this paper, we provide a connection between the local (global) minimizers of non-homogeneous polynomial functions to the maximal (maximum) cliques of hypergraphs whose edges containing $r-1$ and r vertices. This connection can be applied to obtain an upper bound on the Turán densities of complete $\{r-1, r\}$-type hypergraphs.

1. Definitions and introduction

A hypergraph is a pair $H=(V,E)$ of sets such that every element in set E is a subset of set V. V is called the set of vertices and E is called the set of edges. For each hypergraph H, let $e(H)$ denote the number of edges of H. Call $T(H)=\{|e|: e\in E(H)\}$ the *edge type* of a hypergraph H. If $T(H)=\{r_1, r_2, \dots, r_l\}$, then we say that H is an *$\{r_1, r_2, \dots, r_l\}$-hypergraph*. If $l=1$, then we call H r_1-uniform hypergraph or r_1-graph. If $l\geq 2$, then we call H non-uniform hypergraph. For any $r\in T(H)$, the *rth level hypergraph* H^r is the r-uniform hypergraph consisting of all edges of H with r vertices. For a positive integer r, let $V^{(r)}$ be the family of all r-subsets of V, $\overline{H}^r = V^{(r)}\setminus H^r$. Let $\overline{H} = \bigcup_{r\in T(H)}(V^{(r)}\setminus H^r)$. Let K_n^T denote the *complete hypergraph* containing all r-subsets of the vertex set V with n vertices for every $r\in T$. A hypergraph G is called a *subhypergraph* of a hypergraph H, denoted by $G\subseteq H$ if $V(G)\subseteq V(H)$ and $E(G)\subseteq E(H)$. Let $U\subseteq V(H)$, then the subhypergraph of H induced by U, denoted by $H[U]$, has vertex set U and the edge set $E(H[U])=\{e: e\in E(H) \text{ and } e\subseteq U\}$. A complete subhypergraph of a hypergraph H having the same edge type as H is called a clique of H. For any integer $n\in\mathbb{N}$ we denote the set $\{1,2,\dots,n\}$ by $[n]$. We assume that all hypergraphs or graphs have the vertex set $[n]$ throughout the paper if it is not specified and an edge $e=\{i_1, i_2, \dots, i_r\}$ will be denoted by $i_1 i_2 \dots i_r$.

For a non-uniform hypergraph H on n vertices, the Lubell function of H is defined to be

$$h_n(H) = \sum_{r\in T(H)} \frac{|E(H^r)|}{\binom{n}{r}}.$$

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Given a hypergraph F with edge-type T, Johnston and Lu in [5] defined the Turán density of F as

$$\pi(F) = \lim_{n\to\infty} \max\{h_n(H) : |V(H)| = n, H \subseteq K_n^T \text{ and } F \text{ is not isomorphic to } H' \text{ for any } H' \subseteq H\}.$$

Determining the Turán density of a hypergraph in general has been a very challenging problem. Very few results are known and a survey by Keevash on this topic for uniform hypergraphs can be found in [6].

An important tool for Turán type problems is the Lagrangian method. For an r-uniform hypergraph H with n vertices and $\vec{x} \in \mathbb{R}^n$, the *Lagrange function* of H, $\lambda(H, \vec{x})$, is

$$\lambda(H, \vec{x}) := \sum_{e\in E(H)} \prod_{i\in e} x_i.$$

The *standard simplex* of $\mathbb{R}^n$ is denoted by $\Delta := \{\vec{x} \in \mathbb{R}^n : \sum_{i=1}^n x_i = 1, x_i \geq 0 \text{ for every } i \in [n]\}$. A vector $\vec{x} \in \mathbb{R}^n$ is a *feasible weighting* of H if $\vec{x} \in \Delta$. The *Lagrangian* of H is then defined to be

$$\lambda(H) = \max\{\lambda(H, \vec{x}) : \vec{x} \in \Delta\}.$$

Throughout the paper, $\vec{x}$ sometimes refers to a variable and sometimes refers to a fixed vector.

The *characteristic vector* of a non-empty set $U \subseteq [n]$, denoted by $\vec{x}^U = (x_1^U, x_2^U, \ldots, x_n^U)$, is defined as: $x_i^U = \frac{1}{|U|}$ for $i \in U$ and 0 otherwise.

In 1965, Motzkin and Straus established a connection between the Lagrangian of a graph G and the maximum cliques of G.

Theorem 1.1 ([8]). *If G is a 2-graph in which the largest clique has order t, then*

$$\lambda(G) = \lambda(K_t^{\{2\}}) = \frac{1}{2}\left(1 - \frac{1}{t}\right).$$

Furthermore, $\lambda(G, \vec{x})$ reaches the maximum if $\vec{x}$ is the characteristic vector of a maximum clique of H.

Applying this connection, they gave a proof of Turán's classical result on Turán densities of complete graphs. In the 1980's, Sidorenko [11] and Frankl–Füredi [2] further developed this method for hypergraph Turán problems. More studies on Lagrangians of uniform hypergraphs can be found in [13,6,4]. In [12,10,3,9], there are some attempts to generalize Motzkin–Straus type results to hypergraphs. In [3], a Motzkin–Straus type result for $\{1, r\}$-graphs was given by showing the connection between the maximum clique number and the Lagrangian of a $\{1, r\}$-graph if the order of its maximum clique and the order of its maximum complete $\{1\}$-subgraph are the same. The authors of [9] generalized the concept of the Lagrangian from uniform hypergraphs to non-uniform hypergraphs and obtained a result similar to the Motzkin–Straus theorem (Theorem 1.1) for $\{1, 2\}$-hypergraphs. However, the connection between the Lagrangian and the maximum cliques of a graph cannot be extended to hypergraphs in general. Indeed, estimating the Lagrangian of a hypergraph in general is difficult. In [10], Rota Bulò and Pelillo considered the following non-linear program for an r-uniform hypergraph H:

$$\begin{aligned} &\text{minimize} \quad h_H(\vec{x}) = \lambda(\overline{H}, \vec{x}) + \tau \sum_{i=1}^n x_i^r = \sum_{e\in\overline{H}} \prod_{i\in e} x_i + \tau \sum_{i=1}^n x_i^r \\ &\text{subject to} \quad \vec{x} \in \Delta, \end{aligned} \tag{1}$$

where $\tau \in \mathbb{R}$ and $\lambda(\overline{H}, \vec{x}) = \sum_{e\in\overline{H}} \prod_{i\in e} x_i$ is the Lagrangian of $\overline{H}$.

A *local solution* of (1) is a vector $\vec{x} \in \Delta$ for which there exists a neighborhood $\Gamma(\vec{x})$ of $\vec{x}$ such that $h(\vec{y}) \geq h(\vec{x})$ for all $\vec{y} \in \Gamma(\vec{x})$. A *global solution* is a vector $\vec{x} \in \Delta$ such that $h(\vec{y}) \geq h(\vec{x})$ for all $\vec{y} \in \Delta$.

Note that when $r = 2$ and $\tau = \frac{1}{2}$, minimizing $h_H(\vec{x})$ is equivalent to maximizing $\lambda(H, \vec{x})$. In [10], Rota Bulò and Pelillo obtained the following generalization of the Motzkin–Straus theorem.

Theorem 1.2 ([10]). *Let H be an r-uniform hypergraph on $[n]$ and $0 < \tau \leq \frac{1}{r(r-1)}$ (with strict inequality for $r = 2$). A feasible weighting $\vec{x}$ is a local (global) solution of (1) if and only if it is the characteristic vector of a maximal (maximum) clique of H. If H has a maximum clique of order t, then h_H attains its minimum over Δ at τt^{1-r} and the characteristic vector of a maximum clique is a global solution of (1) (This is true for $r = 2$ and $\tau = \frac{1}{2}$ as well).*

This result can be applied to give an upper bound of Turán densities of complete r-uniform hypergraphs. Our purpose is to develop similar results for non-uniform hypergraphs.

In this paper, we study the following non-linear program for an $\{r-1, r\}$-hypergraph H:

$$\begin{aligned} &\text{minimize} \quad f_H(\vec{x}) = \alpha \sum_{e\in\overline{H}^{r-1}} \prod_{i\in e} x_i + \beta \sum_{e\in\overline{H}^{r}} \prod_{i\in e} x_i + \gamma \sum_{i=1}^n x_i^r \\ &\text{subject to} \quad \vec{x} \in \Delta. \end{aligned} \tag{2}$$

The case of $r = 2$ is obtained in [1], so we consider all integers $r \geq 3$. In order to simplify the notation we sometimes write $f_H(\vec{x})$ as $f(\vec{x})$. Let $f_H = \min_{\vec{x}\in\Delta} f(\vec{x})$.

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