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Note

Color-blind index in graphs of very low degree[☆]

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ABSTRACT

Let $c : E(G) \to [k]$ be an edge-coloring of a graph G, not necessarily proper. For each vertex v, let $\bar{c}(v) = (a_1, \ldots, a_k)$, where a_i is the number of edges incident to v with color i. Reorder $\bar{c}(v)$ for every v in G in nonincreasing order to obtain $c^*(v)$, the color-blind partition of v. When c^* induces a proper vertex coloring, that is, $c^*(u) \neq c^*(v)$ for every edge uv in G, we say that c is color-blind distinguishing. The minimum k for which there exists a color-blind distinguishing edge coloring $c : E(G) \to [k]$ is the color-blind index of G, denoted $\text{dal}(G)$. We demonstrate that determining the color-blind index is more subtle than previously thought. In particular, determining if $\text{dal}(G) \leq 2$ is NP-complete. We also connect the color-blind index of a regular bipartite graph to 2-colorable regular hypergraphs and characterize when $\text{dal}(G)$ is finite for a class of 3-regular graphs.

1. Introduction

Coloring the vertices or edges of a graph G in order to distinguish neighboring objects is fundamental to graph theory. While typical coloring problems color the same objects that they aim to distinguish, it is natural to consider how edge-colorings can distinguish neighboring vertices. For an edge-coloring c using colors $\{1, \ldots, k\}$, the *color partition* of a vertex v is given as $\bar{c}(v) = (a_1, \ldots, a_k)$, where the integer a_i is the number of edges incident to v with color i. The edge-coloring c is *neighbor distinguishing* if $\bar{c}$ is a proper vertex coloring of the vertices of G. The *neighbor-distinguishing index* of G is the minimum k such that there exists a neighbor distinguishing k-edge-coloring of G. Define $c^*(v)$ to be the list $\bar{c}(v)$ in nonincreasing order; call $c^*(v)$ the *color-blind partition* at v, since $c^*(v)$ allows for counting the sizes of the color

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classes incident to v without identifying the colors. The edge-coloring c is *color-blind distinguishing* if c^* is a proper vertex coloring of the vertices of G. The *color-blind index* of G, denoted $\text{dal}(G)$, is the minimum k such that there exists a color-blind distinguishing k-edge-coloring of G.

The neighbor-distinguishing index and color-blind index do not always exist for a given graph G. A graph G has no neighbor-distinguishing coloring if and only if it contains a component containing a single edge [4]. The conditions that guarantee G has a color-blind distinguishing coloring are unclear. When a graph G has no color-blind distinguishing coloring, we say that $\text{dal}(G)$ is undefined or write $\text{dal}(G) = \infty$. Kalinowski, Pilśniak, Przybyło, and Woźniak [9] defined color-blind distinguishing colorings and presented several examples of graphs that have no color-blind distinguishing colorings. All of the known examples that fail to have color-blind distinguishing colorings have minimum degree at most three.

When two adjacent vertices have different degrees, their color-blind partitions are distinct for every edge-coloring. Thus, it appears that constructing a color-blind distinguishing coloring is most difficult when a graph is regular and of small degree. Most recent work [1,11] has focused on demonstrating that $\text{dal}(G)$ is finite and small when G is a regular graph (or is almost regular) of large degree. These results were improved by Przybyło [12] in the following theorem.

Theorem 1 (*Przybyło [12]*)**.** *If G is a graph with minimum degree $\delta(G) \geq 3462$, then* $\text{dal}(G) \leq 3$.

We instead focus on graphs with very low minimum degree. In Section 2, we demonstrate that it is difficult to determine $\text{dal}(G)$, even when it is promised to exist.

Theorem 2. *Determining if* $\text{dal}(G) = 2$ *is* NP*-complete, even under the promise that* $\text{dal}(G) \in \{2, 3\}$.

The hardness of determining $\text{dal}(G)$ implies that there is no efficient characterization of graphs with low color-blind index (assuming $\mathsf{P} \neq \mathsf{NP}$). Therefore, we investigate several families of graphs with low degree in order to determine their color-blind index. For example, it is not difficult to demonstrate that $\text{dal}(G) \leq 2$ when G is a tree on at least three vertices.

A 2-regular graph is a disjoint union of cycles, and the color-blind index of cycles is known [9], so we pursue the next case by considering different classes of 3-regular graphs, and determine if they have finite or infinite color-blind index. If G is a k-regular bipartite graph, then the color-blind index of G is at most 3 [9]. In Section 3, we demonstrate that a k-regular bipartite graph has color-blind index 2 exactly when it is associated with a 2-colorable k-regular k-uniform hypergraph. Then using a result of Thomassen [13] and Henning and Yeo [8], this determines the color-blind index of k-regular bipartite graphs when $k \geq 4$.

Theorem 3. *If G is a k-regular bipartite graph where $k \geq 4$, then* $\text{dal}(G) = 2$.

Thus, for k-regular bipartite graphs it is difficult to distinguish between color-blind index 2 or 3 only when $k = 3$.

To further investigate 3-regular graphs, we consider graphs that are very far from being bipartite in Section 4. In particular, we consider a connected 3-regular graph G where every vertex is contained in a 3-cycle. If there is a vertex in three 3-cycles, then G is isomorphic to K_4 and there does not exist a color-blind distinguishing coloring of G [9]. If v is a vertex in two 3-cycles, then one of the neighbors u of v is in both of those 3-cycles. These two 3-cycles form a *diamond*. We say G is a *cycle of diamonds* if G is a 3-regular graph where every vertex in G is in a diamond; G is an *odd cycle of diamonds* if G is a cycle of diamonds and contains $4t$ vertices for an odd integer t. In particular, we consider K_4 to be a cycle of one diamond.

Theorem 4. *Let G be a connected 3-regular graph where every vertex is in at least one 3-cycle of G. Then G has a color-blind distinguishing coloring if and only if G is not an odd cycle of diamonds. When G is not an odd cycle of diamonds, then* $\text{dal}(G) \leq 3$.

2. Hardness of computing dal(G)

In this section, we prove Theorem 2 in the standard way. For basics on computational complexity and NP-completeness, see [3]. It is clear that a nondeterministic algorithm can produce and check that a coloring is color-blind distinguishing, so determining $\text{dal}(G) \leq k$ is in NP. We define a polynomial-time reduction[3] that takes a boolean formula in conjunctive normal form where all clauses have three literals and output a graph with color-blind index two if and only if the boolean formula is satisfiable.

Theorem 2. *Determining if* $\text{dal}(G) = 2$ *is* NP*-complete, even under the promise that* $\text{dal}(G) \in \{2, 3\}$.

Proof. To prove hardness we will demonstrate a polynomial-time reduction that, given an instance ϕ of 3-SAT, will produce a graph G_ϕ such that $2 \leq \text{dal}(G_\phi) \leq 3$ and such that $\text{dal}(G_\phi) = 2$ if and only if ϕ is satisfiable.

Let $\phi(x_1, \dots, x_n) = \bigwedge_{i=1}^{m} C_i$ be a 3-CNF formula with n variables $x_1, \dots, x_n$ and m clauses $C_1, \dots, C_m$. Let each clause C_j be given as $C_j = \hat{x}_{i_{j,1}} \vee \hat{x}_{i_{j,2}} \vee \hat{x}_{i_{j,3}}$, where each $\hat{x}_{i_{j,k}}$ is one of $x_{i_{j,k}}$ or $\neg x_{i_{j,k}}$.

[3] This reduction could easily be implemented in logspace.