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Note

The packing chromatic number of the infinite square lattice is between 13 and 15

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ABSTRACT

Using a SAT-solver on top of a partial previously-known solution we improve the upper bound of the packing chromatic number of the infinite square lattice from 17 to 15. We discuss the merits of SAT-solving for this kind of problem as well as compare the performance of different encodings. Further, we improve the lower bound from 12 to 13 again using a SAT-solver, demonstrating the versatility of this technology for our approach.
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1. Introduction

The notion of packing colouring comes from the area of frequency planning in wireless networks, and was introduced by Goddard et al. in [10] under the name *broadcast colouring*. A *packing k-colouring* of a graph G is a partition of $V(G)$ into disjoint sets $X_1, \ldots, X_k$, so that, for each $i \in \{1, \ldots, k\}$ and $x, y \in X_i$, the minimum distance between x and y in G, $d_G(x, y)$, is greater than i. In other words, vertices with the same colour i are pairwise at distance greater than i. The *packing chromatic number* of a graph G, denoted by $\chi_p(G)$, is the smallest integer k so that there exists a packing k-colouring. A *packing colouring* is a packing k-colouring, for some k, and we sometimes drop the descriptor "packing", when it is clear from the context, to talk simply of a (k-)colouring. Packing colourings have application in frequency planning where one might imagine a broadcast at a higher wavelength travelling further, thus retransmission towers would not be required at such proximity as those for lower wavelengths (see [10]). Broadcast colouring seems to have been renamed packing colouring in the work [1].

The *Cartesian product* of two graphs G and H, denoted $G\Box H$, is the graph with vertex set $G \times H$ and edge set

$$\{((x_1, y_1), (x_2, y_2)) : (x_1 = x_2 \wedge (y_1, y_2) \in E(H)) \vee (y_1 = y_2 \wedge (x_1, x_2) \in E(G))\}.$$

The infinite square lattice (grid) $P_{\mathbb{Z}}\Box P_{\mathbb{Z}}$ is the graph with vertex set $\mathbb{Z} \times \mathbb{Z}$ and edge set

$$\{((x_1, y_1), (x_2, y_2)) : (x_1 = x_2 \wedge |y_1 - y_2| = 1) \vee (y_1 = y_2 \wedge |x_1 - x_2| = 1)\}.$$

If $P_{\mathbb{Z}}$ is the graph with vertices $\mathbb{Z}$ and edges (x, y) given by $|x - y| = 1$, then the infinite square lattice is the product $P_{\mathbb{Z}}\Box P_{\mathbb{Z}}$, which explains our notation.

The packing chromatic number of the infinite square lattice, $\chi_p(P_{\mathbb{Z}}\Box P_{\mathbb{Z}})$, has been the topic of a number of papers. Goddard et al. showed in [10] that $\chi_p(P_{\mathbb{Z}}\Box P_{\mathbb{Z}})$ is finite, more precisely between 9 and 23 (inclusive). In contrast, the packing chromatic

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number of the infinite triangular lattice is infinite [7], though the packing chromatic number of the infinite hexagonal lattice is 7 [16]. The upper bound of [10] is witnessed by a finite grid of dimension $m \times m$ which can be endlessly translated up–down and left–right in order to periodically cover the plane. We call this a *periodic* packing colouring. Such a periodic packing colouring may be seen as a packing colouring of the product $C_m \square C_m$, where C_m is the undirected m-cycle.

Fiala and Lidický [6] then improved the lower bound to 10, and Schwenk [20] improved the upper bound to 22. Later, Ekstein, Fiala, Holub and Lidický used a computer to improve the lower bound to 12 [4] and Soukal and Holub used a clever Simulated Annealing algorithm to improve the upper bound to 17 [23]. Thus these last bounds, in contrast to those that went before, both made fundamental use of mechanical computation.

Further related work. We are not the first to use SAT-solvers in Discrete Mathematics, especially at the interface between Combinatorics and Number Theory. Fascinating progress has been made towards the computation of van der Waerden [3,12] and Ramsey [8] numbers (see also the thesis [17]). Indeed, the case $c = 2$ of the Erdös Discrepancy Conjecture has been settled using this technology [15]. Furthermore, we are not the first to use SAT-solving techniques in packing colouring [22] (though we were not aware of this article when we obtained our results). In [22], the authors translate questions of packing colouring for various finite graphs (including grids) to SAT problems and instances of Integer Programming. The paper [21], by the same authors, which relates similar techniques for other graph colouring problems, should also be mentioned here.

We note some recent contributions in the situation in which packing distances considered for the colours may be specified individually (whereas for us the packing distance and the colour number coincide). This is the situation in [11] and [9]. Research into such alternative packing colourings has gone recently in a more combinatorial direction, see the recent paper [2] for a discussion.

A number of works, both older and new, address questions of computational complexity for determining packing chromatic numbers. It is known that determining the packing chromatic number for general graphs is NP-hard [10]; indeed this remains NP-hard even for trees [5]! Regarding packing chromatic number for Cartesian products of cycles, we should additionally mention the work of [14].

Our story. The first author heard of this problem at a talk by Bernard Lidický at the 8th Slovenian Conference on Graph Theory (Bled 2011). While this problem may not be especially important, few who worked on it can doubt that it is very addictive, and further provides a vehicle through which to ponder different algorithmic techniques. One of the curiosities of the problem is that we have little theoretical insight into it. Note, however, that it is not possible to cover asymptotically more than half of the vertices of the infinite square lattice with the colour 1. Now, suppose there is a packing colouring for the infinite square lattice involving k colours:

- does there exist an m together with an $m \times m$ grid that witnesses a periodic packing k-colouring?
- does there exist a packing k-colouring that has colour 1 at maximal density (1/2) asymptotically?
- does there exist a packing k-colouring so that, for $i < j \leq k$ the asymptotic frequency of colour i is no more than the asymptotic frequency of j?

The answers to the above questions are still not known (though we know the answer to the first question is affirmative, for $k \geq 15$).

Our contribution. In the present note we improve the upper bound from 17 to 16 and then to 15. As with all these upper bounds we give a packing colouring based on a finite grid which can be translated up-down and left–right to give a periodic packing colouring of the infinite square lattice (grid). We make essential use of the periodic 24×24 17-colouring given by Soukal–Holub in [23], which is drawn in Fig. 2.

For our 16-colouring, we take the Soukal–Holub colouring and remove colours 8 to 17, then we blow this up from 24×24 to 48×48 by taking four copies of it (2×2 in shape). We then give the resulting partially coloured grid to a SAT-solver to see if a 16-colouring is possible, which it turns out it is. Our 16-colouring is specified as the obvious periodic translation of the colouring in Fig. 1. Note that our method with a SAT-solver does not run efficiently unless several colours are *planted*, that is some entries (vertices) are preset to a certain colour. For example, with just colour 1 planted at maximal density of 1/2 on a 24×24 grid, the computation runs, on the machines we are using, indefinitely. Planting colours may be seen as a form of *pre-colouring*, thus the objects given to the SAT-solver are partially (pre-)coloured grids. Planting colours allows us also to break the problem's symmetries.

For our 15-colouring, we take the Soukal–Holub colouring and remove colours 6 to 17, then blow this up from 24×24 to 72×72. This colouring is depicted in Fig. 4.

We have additionally improved the lower bound by ruling out the possibility of a certain 12-colouring on a 14×14 grid. Here we derive our result by producing a SAT instance that is found to be unsatisfiable.

2. Encoding and computation

Let $[n] := \{1, \ldots, n\}$. Our *basic encoding*, for investigating the upper bound, involves variables $P_{i,j,k}$, whose being true asserts that position (i, j) in some m-colouring, periodic on a grid of size $n \times n$, is set to colour k. We thus need big clauses of the form $P_{i,j,1} \vee P_{i,j,2} \vee \cdots \vee P_{i,j,m}$ for each $(i, j) \in [n]^2$, together with constraints $\neg P_{i,j,k} \vee \neg P_{i',j',k}$ whenever the distance between (i, j) and (i', j'), $d((i, j), (i', j'))$, is less than k. This distance must, of course be calculated toroidally, i.e.

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