Discrete Applied Mathematics 221 (2017) 18–24

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Communication

On optimal approximability results for computing the strong metric dimension

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ARTICLE INFO

Article history:
Received 22 October 2016
Received in revised form 10 December 2016
Accepted 27 December 2016
Available online 18 January 2017
Communicated by Fred Roberts

Keywords:
Strong metric dimension
Minimum node cover
Approximability
Unique games conjecture
Exponential time hypothesis
Parameterized complexity

ABSTRACT

The strong metric dimension of a graph was first introduced by Sebö and Tannier (2004) as an alternative to the (weak) metric dimension of graphs previously introduced independently by Slater (1975) and by Harary and Melter (1976), and has since been investigated in several research papers. However, the exact worst-case computational complexity of computing the strong metric dimension has remained open beyond being NP-complete. In this communication, we show that the problem of computing the strong metric dimension of a graph of n nodes admits a polynomial-time 2-approximation, admits a $O^*\left(2^{0.287\,n}\right)$-time exact computation algorithm, admits a $O\left(1.2738^k + n\,k\right)$-time exact computation algorithm if the strong metric dimension is at most k, does not admit a polynomial time $(2-\varepsilon)$-approximation algorithm assuming the unique games conjecture is true, does not admit a polynomial time $(10\sqrt{5}-21-\varepsilon)$-approximation algorithm assuming P$\neq$ NP, does not admit a $O^*\left(2^{o(n)}\right)$-time exact computation algorithm assuming the exponential time hypothesis is true, and does not admit a $O^*\left(n^{o(k)}\right)$-time exact computation algorithm if the strong metric dimension is at most k assuming the exponential time hypothesis is true.

1. Introduction

The concept of the metric dimension of graphs was originally introduced independently by Slater [21] and by Harary and Melter [10] in the 1970s. Their definition involved determining a minimum number of nodes such that distance vectors from each of these nodes to all other nodes (the "resolving vectors") can be used to "distinguish" every pair of nodes in the graph. Computing the metric dimension is known to be NP-complete [9]. Optimal approximability results for the metric dimension was provided by Hauptmann et al. in [11] by showing both a $(\ln n + \ln \log_2 n + 1)$-approximation based on an approximation algorithm for test set problems in [2] and also a $(1-\varepsilon)$-inapproximability for any constant $0 < \varepsilon < 1$.

Unfortunately, the metric dimension of a graph suffers from two difficulties, namely that the problem does not provably admit a better-than-logarithmic approximation and the resolving vectors cannot be used to uniquely identify the graph. The **strong** metric dimension of a graph was therefore introduced by Sebö and Tannier [20] as an alternative to the above-mentioned metric dimension of graphs. The resulting "strongly" resolving vectors can indeed be used to uniquely identify the given graph. Subsequently, the strong metric dimension has been investigated in several research papers such as [18,19,25]. Let $G = (V, E)$ be a given undirected graph of n nodes. To define the strong metric dimension, we will use the following notations and terminologies:

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http://dx.doi.org/10.1016/j.dam.2016.12.021

- $\mathrm{N}(u) = \left\{ v \mid \{u, v\} \in E \right\}$ denotes the set of neighbors of a node u.
- $u \overset{s}{\leftrightsquigarrow} v$ denotes a shortest path from between nodes u and v of length (number of edges) $d_{u,v}$.
- $\mathrm{diam}(G) = \max_{u,v \in V} \left\{ d_{u,v} \right\}$ denotes the diameter of a graph G.
- A shortest path $u \overset{s}{\leftrightsquigarrow} v$ is called *maximal*[1] if and only if it is not *properly* included inside another shortest path, *i.e.* , if and only if the predicate

$$\Big(\forall\, x \in \mathrm{N}(u) : d(x, v) \le d(u, v) \Big) \bigwedge \Big(\forall\, y \in \mathrm{N}(v) : d(y, u) \le d(u, v) \Big)$$

  is true.
- A node x *strongly resolves* a pair of nodes u and v, denoted by $x \blacktriangleright \{u, v\}$, if and only if either v is on a shortest path between x and u, or u is on a shortest path between x and v.
- A set of nodes $V' \subseteq V$ is a *strongly resolving set* for G, denoted by $V' \blacktriangleright G$, if and only if every distinct pair of nodes of G is strongly resolved by some node in V'.

Then, the problem of computing the strong metric dimension of a graph can be defined as follows:

Problem name:	Strong Metric Dimension (Str-Met-Dim)
Instance:	an undirected graph $G = (V, E)$.
Valid Solution:	a set of nodes $V' \subseteq V$ such that $V' \blacktriangleright G$.
Objective:	*minimize* $\lvert V' \rvert$.
Related notation:	$\mathrm{sdim}(G) = \min\limits_{V' \subseteq V \,\wedge\, V' \blacktriangleright G} \left\{ \lvert V' \rvert \right\}$.

1.1. Standard concepts from the algorithms research community

For the benefit of readers not familiar with analysis of approximation algorithms, we state below some standard definitions; see standard textbooks such as [8,9,23] for further details. An algorithm for a minimization problem is said to have an *approximation ratio* of ρ (or simply called a ρ-approximation) provided the algorithm runs in polynomial time in the size of the input and produces a solution with an objective value *no larger than* ρ times the value of the optimum. A computational problem P is said to be ρ-inapproximable under a complexity-theoretic assumption of $\mathbb{A}$ provided, assuming $\mathbb{A}$ to be true, there exists no ρ-approximation for P. The (standard) Boolean satisfiability problem when every clause has exactly k literals will be denoted by k- Sat. Finally, for two functions $f(n)$ and $g(n)$ of n, we say $f(n) = O^*(g(n))$ if $f(n) = O(g(n)\, n^c)$ for some positive constant c.

1.2. Brief overview of three well-known complexity theoretic assumptions

For the benefit of those readers not well familiar with well-known complexity-theoretic assumptions, we provide a very brief overview of the three complexity-theoretic assumptions used in this communication.

The P≠NP assumption Starting with the famous Cook's theorem [4] in 1971 and Karp's subsequent paper in 1972 [14], the P≠NP assumption is the central assumption in structural complexity theory and algorithmic complexity analysis.

The Unique Games Conjecture (Ugc) The Unique Games Conjecture, formulated by Khot in [15], is one of the most important open question in computational complexity theory. Informally speaking, the conjecture states that, assuming P≠NP, a type of constraint satisfaction problems does not admit a polynomial time algorithm to distinguish between instances that are almost satisfiable from instances that are almost completely unsatisfiable. There is a large body of research works showing that the conjecture has many interesting implications and many researchers routinely assume Ugc to prove non-trivial inapproximability results. An excellent survey on Ugc can be found in many places, for example in [22].

The Exponential Time Hypothesis (Eth) In an attempt to provide a rigorous evidence that the complexity of k- Sat increases with increasing k, Impagliazzo and Paturi in [12] formulated the so-called Exponential Time Hypothesis (Eth) in the following manner. Letting $s_k = \inf\left\{ \delta \ : \ \text{there exists } O^*\left(2^{\delta n}\right) \text{ algorithm for solving } k\text{- Sat} \right\}$, Eth states that $s_k > 0$ for all $k \ge 3$, *i.e.* , k- Sat does not admit a sub-exponential time (*i.e.* , of time $O^*\left(2^{o(n)}\right)$) algorithm.[2] Eth has significant implications for worst-case time-complexity of exact solutions of search problems, *e.g.* , see [13,24].

1.3. Our results

Let $G = (V, E)$ be the given graph. It is easy to see following the approach in Khuller et al. [17] that the problem of computing the strong metric dimension $\mathrm{sdim}(G)$ can be reduced to an instance of the (unweighted) set-cover problem giving a $O(\log |V|)$-approximation. In this communication, we show further improved results as summarized by the following theorem.

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1 The end-points of such a path is called a mutually maximally distant pairs of nodes in [20].

2 For two functions $f(x)$ and $g(x)$ of x, $f = o(g)$ provided $\lim_{x \to \infty} f(x)/g(x) = 0$.