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Word length perturbations in certain symmetric presentations of dihedral groups

Michael P. Allocca [a], Jason M. Graham [b,*], Candice R. Price [c], Shannon N. Talbott [d], Jennifer F. Vasquez [b]

[a] *Department of Mathematics and Computer Science, Muhlenberg College, Allentown, PA, USA*
[b] *Department of Mathematics, University of Scranton, Scranton, PA, USA*
[c] *Department of Mathematics, University of San Diego, San Diego, CA, USA*
[d] *Department of Mathematics and Computer Science, Moravian College, Bethlehem, PA, USA*

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ABSTRACT

Given a finite group with a generating subset there is a well-established notion of length for a group element given in terms of its minimal length expression as a product of elements from the generating set. Recently, certain quantities called λ_1 and λ_2 have been defined that allow for a precise measure of how stable a group is under certain types of small perturbations in the generating expressions for the elements of the group. These quantities provide a means to measure differences among all possible paths in a Cayley graph for a group, establish a group theoretic analog for the notion of stability in nonlinear dynamical systems, and play an important role in the application of groups to computational genomics. In this paper, we further expose the fundamental properties of λ_1 and λ_2 by establishing their bounds when the underlying group is a dihedral group. An essential step in our approach is to completely characterize so-called symmetric presentations of the dihedral groups, providing insight into the manner in which λ_1 and λ_2 interact with finite group presentations. This is of interest independent of the study of the quantities λ_1, λ_2. Finally, we discuss several conjectures and open questions for future consideration.

1. Introduction and background

For a finite group, G, with generating set, S, there is a notion of length for any element $g \in G$: write g as a product of elements (*i.e.*, as a word) from S, using as few generators as possible. Then the length of g is the number of generators appearing in a minimal expression (*i.e.*, smallest word expression) for g. Furthermore, this notion of length provides a related notion of distance, or metric between two elements g, g' of G. From a geometric perspective, this is related to distances along paths of the Cayley graph for G associated with the generating set S.

Let G be a finite group with symmetric generating set $S = \{s_1, s_2, \ldots, s_n\}$, where symmetric means $1 \notin S$ and $s \in S \Rightarrow s^{-1} \in S$. One reason to consider symmetric generating sets is to ensure that a group element and its inverse have the same length. A word from S in G is any expression formed by taking products of elements in S. We refer to the elements of S as letters. Then, for any element $g \in G$, we define the length $l_S(g) \in [0, \infty)$ of g with respect to S to be the

* Corresponding author.
E-mail address: jason.graham@scranton.edu (J.M. Graham).

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minimal number of letters in S for which g can be written as a word from S. Note that $l_S(g) = 0 \Leftrightarrow g = 1$. Recently, the following quantities have been defined [13]:

$$\lambda_1(G, S) := \max_{g\in G,\ s\in S}\{l_S(gsg^{-1})\}, \tag{1}$$

$$\lambda_2(G, S) := \max_{g\in G,\ s,s'\in S}\{l_S(gss'g^{-1})\}. \tag{2}$$

The quantities λ_1, λ_2 serve to precisely address the questions: given a word, what is the effect on its length after either the deletion of one letter (quantified by λ_1), or the replacement of one letter by another distinct letter (quantified by λ_2)? This presents an analogy between how large these measures can be, and the sensitivity of nonlinear dynamical systems to small perturbations in initial conditions, *i.e.*, the so-called "butterfly effect" [13].

There are several initial observations about λ_1 and λ_2 to note. First, there is the bound

$$\lambda_2(G, S) \leq 2\lambda_1(G, S), \tag{3}$$

which holds for any group G and symmetric generating set S, see [13]. Second, the values for λ_1 and λ_2 in two extreme cases, either when G is commutative or when S is as large as possible, can easily be derived:

Suppose that G is a nontrivial commutative group and S is any symmetric generating set. Then, $\lambda_1(G, S) = 1$ and $\lambda_2(G, S) \leq 2$. This follows because $sg = gs$ for any $g \in G, s \in S$ thus giving

$$l_S(gsg^{-1}) = l_S(s) = 1.$$

Therefore, $\lambda_1(G, S) = 1$.

On the other hand,

$$l_S(gss'g^{-1}) = l_S(ss') \leq 2.$$

Thus, we see that $\lambda_2(G, S) \leq 2$, and note that $\lambda_2(G, S)$ may be equal to 1 (for example, in the case that $ss' \in S$ for all $s, s' \in S$).

Now suppose that G is a nontrivial finite group and $S = G - \{1\}$. That is, we take as our symmetric generating set all elements of G except the identity. Then, for any $g \in G, s \in S$ we have that $g^{-1}sg = h$ for some $h \in G$. Now, either $h = 1$ or $h \in S$. Thus, either $l_S(g^{-1}sg) = 0$ or 1. Thus, we have that

$$\lambda_1(G, S) = \max_{g\in G, s\in S} l_S(g^{-1}sg) = 1,$$

whenever $S = G - \{1\}$. The same basic argument can be used to show that $\lambda_2(G, S) = 1$ whenever $S = G - \{1\}$. Note that this example exhibits the feature that a large generating set S gives small values of λ_1 and λ_2.

Finally, the main result of [13], Theorem 1 from that paper, establishes bounds on λ_1 and λ_2 in the cases where G is the symmetric group Σ_n of order $n!$ and S is one of three distinct generating sets, the transpositions, the reversals, and the Coxeter generators. This together with the aforementioned observations already shows that the values of λ_1, λ_2 are highly dependent on the specific choice of, and specific properties of, both the group G and the particular symmetric generating set S.

We note that an important source of motivation for the definitions of the λ_i, $i = 1, 2$, comes from computational approaches to the study of genome rearrangements. For some time now, combinatorial methods and finite groups such as permutation groups have played a major role in the modeling and exploration of problems arising in combinatorial genomics, for more on such topics see [1,2,8,10,13]. The significance in relation to [13] is that the notion of distance described there relates to the evolutionary distance between species based on differences in their respective genomes.

In addition, it is the case that many groups, including Σ_n, may be described as finitely presented groups [5,12,14]. Furthermore, the so-called group of circular permutations, which is of particular relevance to computational genomics [8], can be described by adding a relation to the usual presentation for the affine symmetric group. Thus, when G is a finitely presented group, there is interest in computing the values for λ_i, $i = 1, 2$ and investigating questions such as how do the relations in a presentation for G affect the values of the λ_i, $i = 1, 2$. To date, very few efforts have been made in this direction.

In this work, we compute bounds or values for λ_1 and λ_2 for the dihedral groups D_n, $n > 2$, which are of course well known to be noncommutative groups of order $2n$, see *e.g.* [7,14]. Specifically, we consider dihedral groups given as finitely presented groups for several small generating sets that have the additional feature of being symmetric as described below, and also examine how the λ_i values vary in both n and the nature of the presentation. First, this involves developing a thorough understanding of the possible symmetric presentations for D_n, which we do completely for symmetric generating sets of cardinality less than or equal to three. Next, it must be established how the relations affect computing values or bounds for the λ_i. An important step along the way is to understand when two or more particular concrete realizations of a presentation in terms of specific elements of D_n correspond to the same abstract presentation.

There are several reasons for considering the values of λ_1 and λ_2 for the dihedral groups. First, some computations are amenable to a direct approach which help to provide insight into certain aspects of the quantities λ_1 and λ_2. For example, when computing the values of λ_1 and λ_2 one is essentially concerned with the lengths of conjugates of elements of S and conjugates of products of pairs of elements of S. That is, we want to know the lengths of elements of the form $g^{-1}sg$ and $g^{-1}ss'g$ where $g \in G$, $s, s' \in S$. Now, it is not necessarily the case that this produces every element of the group.