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Restricted optimal pebbling and domination in graphs

Mustapha Chellali [a], Teresa W. Haynes [b,c,*], Stephen T. Hedetniemi [d], Thomas M. Lewis [e]

[a] LAMDA-RO Laboratory, Department of Mathematics, University of Blida, B.P. 270, Blida, Algeria
[b] Department of Mathematics and Statistics, East Tennessee State University, Johnson City, TN 37614-0002, USA
[c] Department of Mathematics, University of Johannesburg, Auckland Park, 2006, South Africa
[d] Professor Emeritus of Computer Science, School of Computing, Clemson University, Clemson, SC 29634, USA
[e] Department of Mathematics, Furman University, Greenville, SC, USA

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ABSTRACT

For a graph $G = (V, E)$, we consider placing a variable number of pebbles on the vertices of V. A pebbling move consists of deleting two pebbles from a vertex $u \in V$ and placing one pebble on a vertex v adjacent to u. We seek an initial placement of a minimum total number of pebbles on the vertices in V, so that no vertex receives more than some positive integer t pebbles and for any given vertex $v \in V$, it is possible, by a sequence of pebbling moves, to move at least one pebble to v. We relate this minimum number of pebbles to several other well-studied parameters of a graph G, including the domination number, the optimal pebbling number, and the Roman domination number of G.

1. Introduction

Let $G = (V, E)$ be a graph. Let $f : V \to \mathcal{N}$ be a function that assigns to each vertex $v \in V$ a nonnegative integer $f(v) \in \mathcal{N}$. We say that v has been assigned $f(v)$ *pebbles*. Let $w(f) = \sum_{v \in V} f(v)$ equal the total number of pebbles assigned by the function f and that f is a *pebbling configuration*. A *pebbling move* consists of removing two pebbles from a vertex $u \in V$ and then adding one pebble to an adjacent vertex $v \in N(u)$. A pebbling configuration f is said to be *solvable* if for every vertex v, there exists a sequence (possibly empty) of pebbling moves that results in a pebble on v.

The following definition appears in many papers in graph pebbling. The *pebbling number* $\pi(G)$ equals the minimum number k such that every pebbling configuration $f : V \to \mathcal{N}$ with $w(f) = k$ is solvable. Thus, the central focus of graph pebbling is to determine a minimum number of pebbles so that no matter how they are placed on the vertices of a graph G, there will always be a sequence of pebbling moves that can move at least one pebble to any specified vertex of G. Consider a path P_n of order $n \geq 1$. It is easy to see that if all pebbles are initially placed on one of the two end vertices of P_n, then 2^{n-1} pebbles would be required. Thus, pebbling numbers can be exponential in the order n of a graph G.

The concept of pebbling was introduced in the literature by Chung in [3], where she proved that the pebbling number of the n-cube equals 2^n. This result was used to give an alternate proof of a number theoretic theorem of Lemke and Kleitman [10]. Other applications of graph pebbling might include transportation of material. For example, in percolation theory in physics one considers pouring a liquid through a porous substrate. In the process of doing this some of the liquid is

* Corresponding author at: Department of Mathematics and Statistics, East Tennessee State University, Johnson City, TN 37614-0002, USA.
E-mail addresses: m_chellali@yahoo.com (M. Chellali), haynes@etsu.edu (T.W. Haynes), hedet@clemson.edu (S.T. Hedetniemi), tom.lewis@furman.edu (T.M. Lewis).

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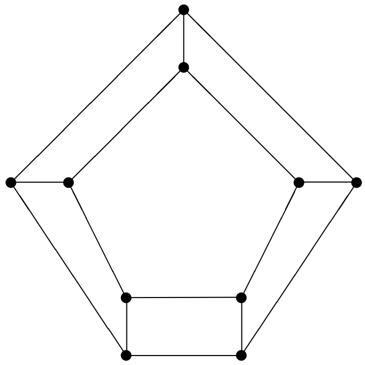

Fig. 1. The graph $G = C_5 \square K_2$.

absorbed by the substrate. One then considers the probability that some liquid will flow all the way through the substrate. The loss of liquid would correspond to discarding one of the two pebbles in a pebbling move; while the liquid flowing through could be measured by covering the vertices. For another example, if a pebbling move is viewed as a transportation problem, one desires to move a unit of some object from a vertex u to an adjacent vertex v with a transportation cost of one unit, e.g. a gallon of gas.

Pachtor et al. [14] defined the *optimal pebbling number* $\pi^*(G)$ to be the minimum weight of a solvable pebbling configuration of G. A solvable pebbling configuration of G with weight $\pi^*(G)$ is called a π^**-configuration*. Optimal pebbling was studied further in [2,5–9,12,13,15,16], for example. The decision problem associated with computing the optimal pebbling number was shown to be NP-Complete in [11].

In this paper, we consider optimal pebbling and introduce a generalization of the optimal pebbling number. We say that a pebbling configuration f is a *t-restricted pebbling configuration (abbreviated t RPC)* if $f(v) \leq t$ for all $v \in V$. We define the *t-restricted optimal pebbling number* $\pi_t^*(G)$ to be the minimum weight of a solvable tRPC on G. If f is a solvable tRPC on G with $w(f) = \pi_t^*(G)$, then f is called a π_t^**-configuration* of G. We note that the limit of t pebbles per vertex applies only to the initial configuration. That is, a pebbling move may place more than t pebbles on a vertex. It follows immediately from the definition that the pebbling configuration which assigns one pebble to every vertex, i.e., $f(v) = 1$ for all $v \in V$, is trivially a solvable tRPC for all $t \geq 1$, and therefore, $\pi_t^*(G)$ is well-defined for all graphs G. Notice also that, by definition, the following inequality sequence holds for any graph G of order n:

$$\pi^*(G) \leq \cdots \pi_t^*(G) \leq \pi_{(t-1)}^*(G) \leq \cdots \pi_2^*(G) \leq \pi_1^*(G) = n.$$

We shall use the following terminology. Let $G = (V, E)$ be a graph of *order* $n = |V|$. The *open neighborhood* of a vertex $v \in V$ is the set $N(v) = \{u \mid uv \in E\}$ of vertices adjacent to v, and its *closed neighborhood* is $N[v] = N(v) \cup \{v\}$. The *open neighborhood of a set* $S \subseteq V$ of vertices is the set $N(S) = \bigcup_{v \in S} N(v)$, while the *closed neighborhood of a set* S is the set $N[S] = \bigcup_{v \in S} N[v]$. The *degree* of a vertex v is $\deg(v) = |N(v)|$. The subgraph of G induced by a set of vertices S is denoted by $G[S]$.

A set $S \subseteq V$ is a *dominating set* of a graph G if every vertex in $V \setminus S$ is adjacent to at least one vertex in S, and S is a *total dominating set* if every vertex in V is adjacent to at least one vertex in S. The *domination number* $\gamma(G)$ (respectively, *total domination number* $\gamma_t(G)$) of a graph G is the minimum cardinality of a dominating set (respectively, total dominating set) in G. A set S is a *connected dominating set* if $G[S]$ is connected, and is a *paired-dominating set* if $G[S]$ has a perfect matching. The *connected domination number* $\gamma_c(G)$ and the *paired-domination number* $\gamma_{pr}(G)$ are defined as expected. A dominating set of G with cardinality $\gamma(G)$ is called a γ*-set* of G. Similar notation is used for the other domination parameters. A *Roman dominating function* on a graph G is a function $f : V \to \{0, 1, 2\}$ satisfying the condition that every vertex u with $f(u) = 0$ is adjacent to at least one vertex v of G for which $f(v) = 2$. The minimum of weight over all such functions is called the *Roman domination number* $\gamma_R(G)$. For any Roman dominating function f of G, and $i \in \{0, 1, 2\}$, let $V_i = \{v \in V(G) \mid f(v) = i\}$. Since the partition $\{V_0, V_1, V_2\}$ defines the function f, we sometimes write $f = \{V_0, V_1, V_2\}$.

Our focus in this paper will be on the 2-restricted optimal pebbling number. As we shall see in Section 2, this parameter is particularly interesting because of its relationship to Roman domination. We also consider relationships between $\pi_2^*(G)$ and other domination parameters, and we characterize graphs having small values for $\pi_2^*(G)$. In Section 3, we give upper bounds on $\pi_2^*(G)$, including one of our main results which shows that for a connected graph G of order $n \geq 2$, $\pi_2^*(G) \leq \lceil 5n/7 \rceil$.

2. 2-restricted optimal pebbling, domination, and roman domination

We begin with an example. The prism $G = C_5 \square K_2$ illustrated in Fig. 1 has $\pi_2^*(G) = 5$. To see this, consider the assignment of 2, 1, 0, 2, 0 pebbles to the vertices in one of the 5-cycles, respectively, and 0 to each of the remaining vertices. It is a simple exercise to see that this assignment is indeed a π_2^*-configuration of G.

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