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Further results on the largest matching root of unicyclic graphs[☆]

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ABSTRACT

Let G be a simple connected graph with vertex set $V(G)$. The matching polynomial of G is defined as $M_G(x) = \sum_{k=0}^{n/2}(-1)^k m(G,k)x^{n-2k}$, where $m(G,k)$ denotes the number of ways in which k independent edges can be selected in G. Let $\lambda_1(G)$ be the largest root of $M_G(x)$. We determine the unicyclic graphs with the four largest and the two smallest $\lambda_1(G)$-values.

1. Introduction

All graphs we consider in this paper are finite and simple. Let $G = (V(G), E(G))$ be a graph with n vertices and m edges. Let $\Gamma(u)$ denote the neighbor set of the vertex u of G. The degree of u in G is denoted by $d_G(u)$, which is equal to $|\Gamma(u)|$. A k-matching in G is a set of k pairwise non-incident edges and the number of k-matchings in G is denoted by $m(G,k)$. The original definition of the matching polynomial is [11] $\sum_k m(G,k)x^k$. However, it is nowadays customary [7,14,15,17] to define this polynomial as

$$M_G(x) = \sum_{k=0}^{n/2}(-1)^k m(G,k)x^{n-2k}. \tag{1}$$

For convenience, we set $m(G,0) = 1$. Clearly, $m(G,k) = 0$ if $k > \frac{n}{2}$. The roots of $M_G(x)$ are called the matching roots of G. It was proven in [19] (independently in [22]) that all roots of the matching polynomial of any graph are real numbers. The largest root of $M_G(x)$, denoted by $\lambda_1(G)$, is called the largest matching root of G. It has been proven [12] that, except in the case of the edgeless graphs $\overline{K_n}$, $\lambda_1(G)$ is always positive.

The reason for defining the matching polynomial via Eq. (1) is that in this case it is related to the characteristic polynomial of the adjacency matrix of G. In particular the matching and characteristic polynomials coincide if and only if G is a forest [15]. Moreover, the matching polynomial of any connected graph is a factor of the characteristic polynomial of some tree, see

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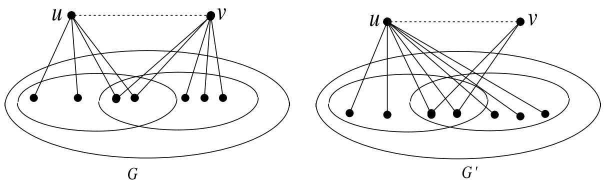

Fig. 1. The Kelmans Transformation.

[14, Theorem 6.1.1]. This provides another way to explain that the roots of matching polynomial are real numbers since the adjacency matrix of any graph is a symmetric matrix and so the roots of its characteristic polynomial are necessarily real-valued.

Graph polynomials and their roots have been much studied in algebraic graph theory (see the recent works [8,10] and the references cited therein). In particular, the matching polynomial, as well as the problems related with its roots, have been studied in due detail [7,14,15,17,21,27,28]. In analogy with the traditional graph–energy concept [16,23], defined to be the sum of the absolute values of the eigenvalues of the adjacency matrix, the matching energy of a graph has been conceived as the sum of the absolute values of the roots of the matching polynomial [18]. This graph invariant has recently attracted much attention; see [1,3–5,24,25] and the references cited therein.

In [12], Fisher and Ryan obtained several bounds for $\lambda_1(G)$. Ghorbani [13] determined the graphs with at most five distinct matching roots. Zhang et al. [27,28] studied $\lambda_1(G)$ of unicyclic graphs and graphs with six distinct matching roots. Other related works can be found in [7,17].

Motivated by the previous research, in this paper, we focus on the ordering unicyclic graphs with respect to the largest matching root. We determine the extremal graphs with the four largest and the two smallest $\lambda_1(G)$.

2. Preliminaries

If $u \in V(G)$, then $G - u$ is the graph obtained from G by deleting the vertex u and the edges of G incident to u. Similarly if $e \in E(G)$, then $G - e$ is the graph obtained from G by deleting the edge e. The following lemmas are well known.

Lemma 2.1 ([7]). *Let $G_1 + G_2$ be the direct sum (disjoint union) of the graphs G_1 and G_2. Then $M_{G_1+G_2}(x) = M_{G_1}(x)M_{G_2}(x)$.*

Lemma 2.2 ([7]). *Let G be a graph and $u \in V(G)$. Suppose the neighborhood of u is $\Gamma(u) = \{v_1, v_2, \ldots, v_d\}$. Then $M_G(x) = xM_{G-u}(x) - \sum_{v_i \in \Gamma(u)} M_{G-uv_i}(x)$.*

Corollary 2.3 ([7]). *If P_n is the path on n vertices, then $M_{P_n}(x) = xM_{P_{n-1}}(x) - M_{P_{n-2}}(x)$.*

Lemma 2.4 ([7]). *Let $u, v \in V(G)$ and $uv \in E(G)$. Then $m(G, k) = m(G - uv, k) + m(G - \{u, v\}, k)$, and therefore $M_G(x) = M_{G-uv}(x) - M_{G-u-v}(x)$.*

Lemma 2.5 ([7]). *Let G^* be a spanning subgraph of G, $\lambda_1(G)$ be the largest matching root of G. If $x \geq \lambda_1(G)$ then $M_{G^*}(x) \geq M_G(x)$. If G^* is a proper subgraph of G and $x > \lambda_1(G)$, then $M_{G^*}(x) > M_G(x)$.*

The following transformation, called Kelmans Transformation, plays an prominent role in the sequel. This transformation is widely used in many other problems [6].

Definition 2.6. Let u, v be two vertices of the graph G. The Kelmans Transformation of G is as follows (cf. Fig. 1): erase all edges between v and $N(v) \setminus (N(u) \cup u)$ and add all edges between u and $N(v) \setminus (N(u) \cup u)$. Let us call u and v the beneficiary and the co-beneficiary of the transformation, respectively. The obtained graph has same number of edges as G; in general we will denote it by G' without referring to the vertices u and v.

Lemma 2.7 ([6,20]). *Assume that G' is a graph obtained from G by some Kelmans transformation. Then $\lambda_1(G') \geq \lambda_1(G)$.*

Lemma 2.8 ([28]). *For two disjoint graphs G_1 and G_2 with $u \in V(G_1)$ and $v \in V(G_2)$, let G be the graph obtained from G_1 and G_2 by adding an edge uv. Let G' be the graph obtained from G_1 and G_2 by identifying u and v (to a new vertex say w), and then adding a pendent edge to w. Then $\lambda_1(G) < \lambda_1(G')$.*

Lemma 2.9 ([27]). *Let $G_{n,k}$ be the graph of order n obtained from a cycle C_k by attaching $n - k$ pendent edges at one vertex of C_k. Then $\lambda_1(G_{n,k}) < \lambda_1(G_{n,k-1})$.*