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$L(2, 1)$-Labeling of Kneser graphs and coloring squares of Kneser graphs

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ABSTRACT

The frequency assignment problem is to assign a frequency to each radio transmitter so that transmitters are assigned frequencies with allowed separations. Motivated by a variation of the frequency assignment problem, the $L(2, 1)$-labeling problem was put forward. An $L(2, 1)$-labeling of a graph G is a function f from the vertex set $V(G)$ to the set of all nonnegative integers such that $|f(x) - f(y)| \geq 2$ if $d(x, y) = 1$ and $|f(x) - f(y)| \geq 1$ if $d(x, y) = 2$, where $d(x, y)$ denotes the distance between x and y in G. The $L(2, 1)$-labeling number $\lambda(G)$ of G is the smallest number k such that G has a $L(2, 1)$-labeling with $\max\{f(v) : v \in V(G)\} = k$. Griggs and Yeh (1992) conjectured that $\lambda(G) \leq \Delta^2$ for any graph with maximum degree $\Delta \geq 2$. The *Kneser graph* $K(a, b)$ is defined as the graph whose vertices correspond to all b-subsets of the a-set $A = \{1, 2, \ldots, a\}$, with edges joining pairs of vertices that correspond to non-overlapping b-subsets. The chromatic number of the Kneser graph $K(a, b)$ is $a-2b+2$. Füredi put forward an open problem: What is the value for the chromatic number of the square of any Kneser graph $K(a, b)$ (the square of a graph is the graph obtained by adding edges joining vertices at distance 2)? In this article, the $L(2, 1)$-labeling numbers of Kneser graphs $K(a, b)$ are considered. The combined upper bounds for the $L(2, 1)$-labeling numbers of Kneser graphs are derived by using two approaches. The exact $L(2, 1)$-labeling number of Kneser graph $K(a, b)$ for $a \geq 3b - 1$ is obtained, and it is proved that Griggs and Yeh's conjecture holds for Kneser graphs and the $L(2, 1)$-labeling numbers of Kneser graphs are much better than Δ^2 in most cases. We also provide bounds for the chromatic number of the square of any Kneser graph $K(a, b)$ using the proof for the upper bounds of the $L(2, 1)$-labeling numbers of Kneser graphs.

1. Introduction

In the frequency assignment problem, radio transmitters are assigned frequencies with some separation in order to reduce interference. This problem can be formulated as a graph coloring problem [11]. Roberts [24] proposed a new version of the frequency assignment problem with two restrictions: radio transmitters that are "close" must be assigned different frequencies; those that are "very close" must be assigned frequencies at least two apart. To formulate the problem in graph theoretic terms, radio transmitters are represented by vertices of a graph; adjacent vertices are considered "very close" and vertices at distance two are considered "close". Motivated by this problem, Griggs and Yeh [10] proposed the following labeling on a simple graph. Let $d(x, y)$ be the distance between vertices x and y in a graph G. An $L(2, 1)$-*labeling* of a graph G

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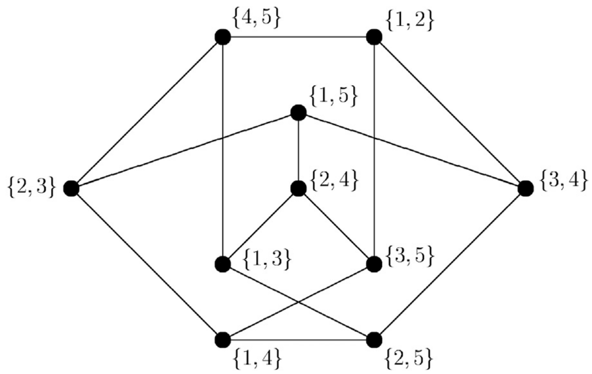

Fig. 1. Petersen graph.

is a function f from all vertices of G to nonnegative integers such that $|f(x) - f(y)| \geq 2$ if $d(x, y) = 1$ and $|f(x) - f(y)| \geq 1$ if $d(x, y) = 2$. For an $L(2, 1)$-labeling, if the maximum label is no greater than k, then it is called a *k-$L(2, 1)$-labeling*. The *$L(2, 1)$-labeling number* of G, denoted by $\lambda(G)$, is the smallest number k such that G has a k-$L(2, 1)$-labeling.

From then on, a large number of articles have been published devoted to the study of the frequency assignment problem and its connections to graph labelings, in particular, to the class of $L(2, 1)$-labelings and its generalizations. In addition to graph theory and combinatorial techniques (see references [1–4,6,9,10,12,14,17,20,23,25–31]), other interesting approaches in studying these labelings include: neural networks [8,18], genetic algorithms [19], and simulated annealing [7,22]. Since graph coloring problems are notoriously difficult, researchers have to consider the problem of obtaining $\lambda(G)$ for particular classes of graphs.

Griggs and Yeh [10] proved that it is NP-complete to decide whether a given graph G allows an $L(2, 1)$-labeling of span at most n. Thus, it is important to obtain good lower and upper bounds for λ. For a diameter two graph G, it is known that $\lambda(G) \leq \Delta^2$, where $\Delta = \Delta(G)$ is the maximum degree of G, and the upper bound can be attained by *Moore graphs*, that is, diameter 2 graphs of order $\Delta^2 + 1$ [10]. Based on the previous research, Griggs and Yeh [10] conjectured that $\lambda(G) \leq \Delta^2$ holds for any graph G with $\Delta \geq 2$. The conjecture is known as the Δ^2-conjecture and considered as the most important open problem in the area. Griggs and Yeh [10] proved originally that $\lambda(G) \leq \Delta^2 + 2\Delta$, the best bound $\Delta^2 + \Delta - 2$ so far is due to Gonçalves [9]. Havet, Reed and Sereni [12] proved that the Δ^2-conjecture holds for sufficiently large Δ.

Both the vertex coloring problem and the $L(2, 1)$-labeling problem are known to be strongly NP-hard. A number of studies have been devoted to the $L(2, 1)$-labeling problem (see references). Due to the inherent hardness of this problem, most papers have to consider only particular classes of graphs.

Kneser graphs are an important graph class which has been extensively studied in the context of coloring problems. A set A of cardinality a is called an a-set. Similarly a subset B of A of cardinality b is called a b-subset of A. Given positive integers a, b with $a \geq b$, the *Kneser graph* $K(a, b)$ is defined as the graph whose vertices correspond to all b-subsets of the a-set $A = \{1, 2, \ldots, a\}$. Two nodes x, y that correspond to b-subsets X and Y of A are joined by an edge if and only if $X \cap Y = \emptyset$. It is easy to see that $K(a, 1) = K_a$, where K_a is the complete graph of a vertices. Note that $K(5, 2)$ is the Petersen Graph (see Fig. 1). Since $K(a, b)$ has no edges when $a < 2b$, we only consider here the case when $a \geq 2b$. In 1978, Lovász ([13] and [21]) proved that $\chi(K(a, b)) = a - 2b + 2$ where $\chi(G)$ denotes the chromatic number of a graph G. This was first conjectured by Kneser in 1955. A number of results have also been obtained about the related concept of circular chromatic numbers of Kneser graphs, see, e.g., [32].

Let G be a simple graph. The square of G, G^2 is defined as follows: the vertex set of G^2 is the same as the vertex set of G; any two distinct vertices of G^2 are adjacent if and only if the distance between them in G is at most 2. For coloring squares of graphs, Füredi put forward an open problem: What is the value for the chromatic number of the square of any Kneser graph $K(a, b)$? In [16], Kim and Park obtained upper bounds for the chromatic number of the square of any Kneser graph $K(2b + 1, b)$.

In this article, we consider the $L(2, 1)$-labeling numbers of Kneser graphs $K(a, b)$. We use two approaches to derive the combined upper bounds for the $L(2, 1)$-labeling numbers of Kneser graphs. We obtain the exact $L(2, 1)$-labeling number of Kneser graph $K(a, b)$ for $a \geq 3b - 1$, and prove that Griggs and Yeh's conjecture holds for Kneser graphs and the $L(2, 1)$-labeling numbers of Kneser graphs are much better than Δ^2 in most cases. We also provide bounds for the chromatic number of the square of any Kneser graph $K(a, b)$ using the proof for the upper bounds of the $L(2, 1)$-labeling numbers of Kneser graphs.

2. A labeling algorithm

A subset X of $V(G)$ is called an *i-stable set* (or *i-independent set*), if the distance between any two vertices in X is greater than i. A 1-stable set is a usual independent set. A *maxima* i-stable subset X of a set Y of vertices is an i-stable subset of Y such that X is not a proper subset of any other i-stable subset of Y.

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