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Cycles embedding in folded hypercubes with conditionally faulty vertices

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ABSTRACT

A network is said to be conditionally faulty if its every vertex is incident to at least g fault-free vertices, where $g \geq 1$. An n-dimensional folded hypercube FQ_n is a well-known variation of an n-dimensional hypercube Q_n, which can be constructed from Q_n by adding an edge to every pair of vertices with complementary addresses. In this paper, we define that a network is said to be g-conditionally faulty if its every vertex is incident to at least g fault-free vertices. Then, let FF_v denote the set of faulty vertices in FQ_n, we consider the cycles embedding properties in 4-conditionally faulty $FQ_n - FF_v$, as follows:

1. For $n \geq 3$, $FQ_n - FF_v$ contains a fault-free cycle of every even length from 4 to $2^n - 2 \mid FF_v \mid$, where $|FF_v| \leq 2n - 5$;
2. For even $n \geq 4$, $FQ_n - FF_v$ contains a fault-free cycle of every odd length from $n + 1$ to $2^n - 2 \mid FF_v \mid -1$, where $|FF_v| \leq 2n - 5$.

1. Introduction

Choosing an appropriate *interconnection network* (*network* for short) is an important integral part of designing parallel processing and distributed systems. There are a large number of network topologies which have been proposed. The interested readers may refer to [2,14,30] for extensive references. Among the proposed network topologies, the *hypercube* [3] is a well-known network model which has several excellent properties, such as recursive structure, regularity, symmetry, small diameter, short mean internode distance, low degree, and much smaller edge complexity, which are very important for designing massively parallel or distributed systems [22]. Numerous variants of the hypercube have been proposed in the literature [5,6,26]. One variant that has been the focus of a great deal of research is the *folded hypercube*, which can be constructed from a hypercube by adding an edge to every pair of vertices that are the farthest apart, i.e., two vertices with complementary addresses. The folded hypercube has been shown to be able to improve the system's performance over a regular hypercube in many measurements, such as diameter, fault diameter, connectivity, and so on [5,28].

An important feature of an interconnection network is its ability to efficiently simulate algorithms designed for other architectures. Such a simulation can be formulated as *network embedding*. An *embedding* of a *guest network* G into a *host network* H is defined as a one-to-one mapping f from the vertex set of G to the vertex set of H. Under f, an edge in G corresponds to a path in H [22]. The embedding strategy allows us to emulate the effect of a guest network on a host network. Then, algorithms developed for a guest network can also be executed well on the host network.

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Cycles (rings), the most fundamental networks for parallel and distributed computation, are suitable for designing simple algorithms with low communication costs. Numerous efficient algorithms designed on rings for solving various algebraic problems and graph problems can be found in [1,22]. Rings can be used as control/data flow structures for distributed computing in arbitrary networks. These applications motivate the embedding of cycles in networks.

Since vertices and/or edges in a network may fail accidentally, it is demanded to consider the fault-tolerance of a network. Hence, the issue of fault-tolerant cycle embedding in an n-dimensional folded hypercube FQ_n has been studied in [4,7,9,10,12,13,20,19,16–18,25,28,31]. Let FF_v and FF_e denote the sets of faulty vertices and faulty edges in FQ_n. In [28], Wang showed that $FQ_n - FF_e$, i.e., removing the edges in FF_e from FQ_n, still has a Hamiltonian cycle of length 2^n when $|FF_e| \leq n-1$. For the case where $|FF_v| = 1$, Hsieh [13] showed that $FQ_n - FF_v$ contains a fault-free cycle of every even length from 4 to $2^n - 2$ if $n \geq 3$, and for even $n \geq 2$, $FQ_n - FF_v$ contains a fault-free cycle of every odd length from $n+1$ to $2^n - 1$. Recently, for even $n \geq 2$, Cheng [4] showed that every fault-free edge of $FQ_n - FF_v$ lies on a cycle of every odd length from $n+1$ to $2^n - 3$, where $|FF_v| = 1$. For the case where $|FF_v| = 1$, Kuo [16] extended Cheng's [4] results to obtain that every fault-free edge of $FQ_n - FF_v$ lies on a cycle of every even length from 4 to $2^n - 2$ if $n \geq 3$, and for even $n \geq 2$, every fault-free edge of $FQ_n - FF_v$ also lies on a cycle of every odd length from $n+1$ to $2^n - 1$. However, one should notice that each component in a network may have independent reliability. That is, if components of a network fail independently, the probability that all failures would be close to each other becomes low. Due to this motivation, Harary [8] first introduced the idea of *conditional connectivity*. Later, Latifi [21] defined the *conditional vertex-faults* which require each vertex of a network is incident to at least g fault-free vertices, $g \geq 1$. For the convenience of description, we define that a network is said to be *g-conditionally faulty* if its every vertex is incident to at least g fault-free vertices. Recently, under the 4-conditionally faulty and $|F_{FQ_n}(e)| \leq n-3$,[1] Kuo [18] showed that $FQ_n - FF_v$ contains a fault-free cycle of every even length from 4 to $2^n - 2|FF_v|$ if $n \geq 4$, and for even $n \geq 4$, $FQ_n - FF_v$ contains a fault-free cycle of every odd length from $n+1$ to $2^n - 2|FF_v| - 1$, where $|FF_v| \leq 2n - 7$. According to the above motivations, we extend Kuo's [18] results to consider the cycles embedding properties in 4-conditionally faulty $FQ_n - FF_v$ of this paper, as follows:

1. For $n \geq 3$, $FQ_n - FF_v$ contains a fault-free cycle of every even length from 4 to $2^n - 2|FF_v|$, where $|FF_v| \leq 2n - 5$;
2. For even $n \geq 4$, $FQ_n - FF_v$ contains a fault-free cycle of every odd length from $n+1$ to $2^n - 2|FF_v| - 1$, where $|FF_v| \leq 2n - 5$.

Throughout this paper, a number of terms – network and graph, node and vertex, edge and link – are used interchangeably. The remainder of this paper is organized as follows: In Section 2, we provide some necessary definitions and notations. We present our main result in Section 3. Some concluding remarks are given in Section 4.

2. Preliminaries

A graph $G = (V, E)$ is an ordered pair in which V is a finite set and E is a subset of $\{(u, v) | (u, v)$ is an unordered pair of $V\}$. We say that V is the *vertex set* and E is the *edge set*. We also use $V(G)$ and $E(G)$ to denote the vertex set and the edge set of G, respectively. Two vertices u and v are *adjacent* if $(u, v) \in E$. For the edge $e = (u, v)$, u and v are called the *end-vertices* of e. We call u adjacent to v, and vice versa. A graph $G = (V_0 \cup V_1, E)$ is bipartite if $V_0 \cap V_1 = \emptyset$ and $E \subseteq \{(x, y) | x \in V_0 \text{ and } y \in V_1\}$. A path $P[v_0, v_k] = \langle v_0, v_1, \ldots, v_k \rangle$ is a sequence of distinct vertices in which any two consecutive vertices are adjacent. We call v_0 and v_k the *end-vertices* of the path. In addition, a path may contain a *subpath*, denoted as $\langle v_0, v_1, \ldots, v_i, P[v_i, v_j], v_j, v_{j+1}, \ldots, v_k \rangle$, where $P[v_i, v_j] = \langle v_i, v_{i+1}, \ldots, v_{j-1}, v_j \rangle$. The length of a path is the number of edges on the path. A path $\langle v_0, v_1, \ldots, v_k \rangle$ forms a *cycle* if $v_0 = v_k$ and $v_0, v_1, \ldots, v_{k-1}$ are distinct. A vertex is *fault-free* if it is not faulty. An edge is *fault-free* if the two end-vertices and the edge between them are not faulty. A path (cycle) is *fault-free* if it contains no faulty edges and faulty vertices. For graph-theoretic terminologies and notations are not mentioned here, readers may refer to [29].

An *n-dimensional hypercube* Q_n (n-cube for short) can be represented as an undirected graph such that $V(Q_n)$ consists of 2^n vertices which are labeled as binary strings of length n from $\underbrace{00\ldots0}_{n}$ to $\underbrace{11\ldots1}_{n}$. Each edge $e = (u, v) \in E(Q_n)$ connects two vertices u and v if and only if u and v differ in exactly one bit of their labels, i.e., $u = b_n b_{n-1} \ldots b_k \ldots b_1$ and $v = b_n b_{n-1} \ldots \bar{b}_k \ldots b_1$, where $\bar{b}_k$ is the *one's complement* of b_k, i.e., $\bar{b}_k = 1 - i$ if $b_k = i$ for $i \in \{0, 1\}$. We call that e is an edge of *dimension* k. Clearly, each vertex connects to exactly n other vertices. In addition, there are 2^{n-1} edges in each dimension and $|E(Q_n)| = n \cdot 2^{n-1}$. Fig. 1 shows a 2-dimensional hypercube Q_2 and a 3-dimensional hypercube Q_3.

Let $x = x_n x_{n-1} \ldots x_1$ and $y = y_n y_{n-1} \ldots y_1$ be two n-bit binary strings; and let $y = x^{(k)}$, where $1 \leq k \leq n$, if $y_k = 1 - x_k$ and $y_i = x_i$ for all $i \neq k$, $1 \leq i \leq n$. In addition, let $y = \bar{x}$ if $y_i = 1 - x_i$ for all $1 \leq i \leq n$. The *Hamming distance* $d_H(x, y)$ between two vertices x and y is the number of different bits in the corresponding strings of the vertices. The *Hamming weight* $hw(x)$ of x is the number of i's such that $x_i = 1$. Note that Q_n is a bipartite graph with two partite sets $\{x | hw(x)$ is odd$\}$ and $\{x | hw(x)$ is even$\}$. Let $d_{Q_n}(x, y)$ be the *distance* between two vertices x and y in graph Q_n. Clearly, $d_{Q_n}(x, y) = d_H(x, y)$.

An *n-dimensional folded hypercube* FQ_n can be constructed from an n-cube by adding an edge (also called *complementary edge*) to every pair of vertices that are the farthest apart, i.e., for a vertex whose address is $b = b_n b_{n-1} \ldots b_1$, it now has one more edge to vertex $\bar{b} = \bar{b}_n \bar{b}_{n-1} \ldots \bar{b}_1$, in addition to its original n edges. Thus, FQ_n has 2^{n-1} more edges than Q_n. We call

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[1] $F_{FQ_n}(e)$ denotes the set of faulty vertices which are incident to the end-vertices of any fault-free edge $e \in E(FQ_n)$.