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Comparing the metric and strong dimensions of graphs

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ABSTRACT

Let G be a graph and u, v be any two distinct vertices of G. A vertex w of G *resolves* u and v if the distance between u and w does not equal the distance between v and w. A set W of vertices of G is a resolving set for G if every pair of vertices of G is resolved by some vertex of W. The minimum cardinality of a resolving set for G is the *metric dimension*, denoted by $\dim(G)$. If G is a connected graph, then a vertex w *strongly resolves* two vertices u and v if there is a shortest u–w path containing v or a shortest v–w path containing u. A set S of vertices is a *strong resolving set* for G if every pair of vertices is strongly resolved by some vertex of S and the minimum cardinality of a strong resolving set is called the *strong dimension* of G and is denoted by $\mathrm{sdim}(G)$. Both the problem of finding the metric dimension and the problem of finding the strong dimension of a graph are known to be NP-hard. It is known that the strong dimension can be polynomially transformed to the vertex covering problem for which good approximation algorithms are known. The metric dimension is a lower bound for the strong dimension. In this paper we compare the metric and strong dimensions of graphs. We describe all trees for which these invariants are the same and determine the class of trees for which the difference between these invariants is a maximum. We observe that there is no linear upper bound for the strong dimension of trees in terms of the metric dimension. For cographs we show that $\mathrm{sdim}(G) \leq 3\dim(G)$ and that this bound is asymptotically sharp. It is known that the problem of finding the metric dimension of split graphs is NP-hard. We show that the strong dimension of connected split graphs can be found in polynomial time.

1. Introduction

Let G be a graph and x, y any two vertices of G. Then the distance between x and y, denoted by $d(x, y)$, is the length of a shortest x–y path in G if such a path exists and it is defined to be ∞ otherwise. A vertex w of G *resolves* two distinct vertices u and v if $d(u, w) \neq d(v, w)$. A set W of vertices of G is a resolving set for G if every pair of vertices of G is resolved by some vertex of W. A smallest resolving set for G is called a *metric basis* and its cardinality the *metric dimension*, denoted by $\dim(G)$. Slater [20] and Harary and Melter [11] independently introduced the metric dimension of a graph. Slater referred to a resolving set as a *locating set* and the smallest cardinality of a locating set as the *location number* of a graph. He motivated the study of locating sets by their application to uniquely determining the location of an intruder in a network. Since then many other applications of resolving sets and the metric dimension have been discussed in the literature, see [3] for an extensive summary of such applications. NP-hardness for the problem of finding the metric dimension was established in [14].

Let $W = \{w_1, w_2, \ldots, w_k\}$ be an (ordered) set of vertices in a graph G and let v be any vertex of G. Then the vector $d(v|W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$ is called the *distance vector* of v with respect to W. Thus W is a resolving set if

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and only if the distance vectors with respect to W of any two distinct vertices are different. Sebő and Tannier [19] observed that a metric basis and the corresponding distance vectors are not sufficient to uniquely describe the graph. This motivated them to introduce a 'stronger' version of the metric dimension of a graph. If G is a connected graph, then a vertex w *strongly resolves* two vertices u and v if there is a shortest u-w path containing v or a shortest v-w path containing u. A set S of vertices is a *strong resolving set* for G if every pair of vertices is strongly resolved by some vertex of S and a smallest strong resolving set is called a *strong basis* and its cardinality the *strong dimension* of G, denoted by $\mathrm{sdim}(G)$. It is pointed out in [19] that a strong basis and the corresponding distance vectors uniquely determine the graph. For a more detailed discussion of this unique determination see [13]. NP-hardness for the problem of finding the strong dimension of a connected graph was established in [17]. In the same paper it was shown that the problem of finding the strong dimension of a graph can be polynomially transformed to the vertex covering problem. To this end, a vertex v of a connected graph is said to be *maximally distant* from a vertex u, denoted by $v \,\mathrm{MD}\, u$, if for all $x \in N(v)$, $d(x, u) \leq d(v, u)$, i.e., the neighbours of v are no further from u than v is from u. If $v \,\mathrm{MD}\, u$ and $u \,\mathrm{MD}\, v$, then u and v are said to be *mutually maximally distant* and this is denoted by $u \,\mathrm{MMD}\, v$. The *strong resolving* graph of a connected graph G, denoted by G_{SR}, has as its vertex set the vertices of G and two vertices u and v are adjacent in G_{SR} if and only if $u \,\mathrm{MMD}\, v$. A set S of vertices of a graph G is a *vertex cover* of the graph if every edge of G is incident with a vertex of S. The cardinality of a smallest vertex cover of G is called the *vertex covering number* and is denoted by $\alpha(G)$. The following result was established in [17].

Theorem 1.1 (*[17]*). *Let G be a connected graph. Then a set S of vertices of G is a strong basis for G if and only if S is a minimum vertex cover of G_{SR}.*

Hence $\alpha(G_{SR}) = \mathrm{sdim}(G)$. Note that the problem of finding a minimum vertex cover for G_{SR} is equivalent to finding a minimum vertex cover for the graph obtained by deleting all isolated vertices from G_{SR}. (Since the isolated vertices of the strong resolving graph do not play a role when finding the minimum vertex cover, these were omitted when discussing the strong resolving graph in [18].) It is known, see [9], that a factor-2 approximation for the vertex covering number can be found in a greedy manner by repeatedly taking both endpoints of an edge into the vertex cover, and then removing these vertices from the remaining graph until no edges remain.

The metric dimension is a lower bound for the strong dimension. In this paper we compare the metric and strong dimensions of several classes of graphs for which the metric dimension can be found in polynomial time. More specifically, we describe all trees for which these invariants are equal and characterize those trees for which the difference between these two invariants is a maximum. We show that there is no linear upper bound for the strong dimension of trees in terms of the metric dimension. For a connected *cograph* G (i.e., a graph without an induced P_4) we determine a lower bound on the metric dimension in terms of its order and establish the asymptotically sharp bound $\mathrm{sdim}(G) \leq 3\dim(G)$. It is known that the problem of finding the metric dimension of *split graphs* (i.e. graphs whose vertex set can be partitioned into a clique and an independent set) is NP-hard [8]. On the other hand we show that the strong dimension of connected split graphs can be found efficiently, thereby giving an upper bound for the metric dimension of these graphs.

For basic graph theory terminology not introduced here we follow [5]. We use n to denote the order of a graph. For vertices u and v of a graph G, $u \sim v$ (or $u \sim_G v$) means that a vertex u is adjacent with a vertex v and $u \nsim v$ (or $u \nsim_G v$) means that u and v are non-adjacent. Let G be a graph and S_1 and S_2 sets of vertices of G. If every vertex of S_1 is MMD from every vertex of S_2 we write $S_1 \,\mathrm{MMD}\, S_2$. Moreover, if no vertex of S_1 is MMD with any vertex of S_2 we denote this by $S_1 \neg\,\mathrm{MMD}\, S_2$. If either S_1 or S_2, say S_1, consists of a single vertex w, then we replace S_1 by w. If we need to specify the graph G in which two vertices are or are not MMD we use MMD_G or $\neg\mathrm{MMD}_G$, respectively. For a graph G, we let $\eta(G) = \mathrm{sdim}(G) - \dim(G)$.

2. Trees

In this section we will be comparing the metric and strong dimension for trees. Efficient processes for finding the metric dimension of trees have been described independently in several papers—see, for example, [4,11,14,20]. Suppose T is a tree. We begin by introducing some terminology introduced in [4] that is useful when describing an algorithm for finding the metric dimension for trees. A vertex of degree at least 3 in a graph G is called a *major vertex* of G. A major vertex v is called an *exterior major vertex* of T if T–v contains at least one component that is a path. We denote the number of exterior major vertices of T by $\mathrm{ex}(T)$.

Let $\sigma(T)$ be the number of leaves in T. In his seminal paper on the metric dimension of a graph, Slater [20] established the following formula for the metric dimension of trees:

Theorem 2.1 (*[20]*).

(i) *If T is a tree that is not a path, then*

$$\dim(T) = \sigma(T) - \mathrm{ex}(T).$$

(ii) *The metric dimension of a non-trivial path is* 1.

Sebö and Tannier [19] observed that the strong dimension for trees has the following simple formula.

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