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Dominating and irredundant broadcasts in graphs

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ABSTRACT

A *broadcast* on a nontrivial connected graph $G = (V, E)$ is a function $f : V \to \{0, 1, \ldots, \text{diam}(G)\}$ such that $f(v) \le e(v)$ (the eccentricity of v) for all $v \in V$. The cost of f is $\sigma(f) = \sum_{v \in V} f(v)$. A broadcast f is *dominating* if each $u \in V$ is at distance at most $f(v)$ from a vertex v with $f(v) \ge 1$.

We use properties of minimal dominating broadcasts to define the concept of an irredundant broadcast on G. We determine conditions under which an irredundant broadcast is maximal irredundant. Denoting the minimum costs of dominating and maximal irredundant broadcasts by $\gamma_b(G)$ and $\text{ir}_b(G)$ respectively, the definitions imply that $\text{ir}_b(G) \le \gamma_b(G)$ for all graphs. We show that $\gamma_b(G) \le \frac{5}{4}\,\text{ir}_b(G)$ for all graphs G.

We also briefly consider the upper broadcast number $\Gamma_b(G)$ and upper irredundant broadcast number $\text{IR}_b(G)$, and illustrate that the ratio IR_b/Γ_b is unbounded for general graphs.

1. Introduction

A *broadcast* on a nontrivial connected graph $G = (V, E)$ is a function $f : V \to \{0, 1, \ldots, \text{diam}(G)\}$ such that $f(v) \le e(v)$ (the eccentricity of v) for all $v \in V$. If G is disconnected, we define a broadcast on G as the union of broadcasts on its components. A broadcast f is *dominating* if each $u \in V$ is at distance at most $f(v)$ from a vertex v with $f(v) \ge 1$. The *cost* of a broadcast f is $\sigma(f) = \sum_{v \in V} f(v)$, and the *broadcast number* of G is

$$\gamma_b(G) = \min\{\sigma(f) : f \text{ is a dominating broadcast of } G\}.$$

A dominating broadcast f such that $f(v) \in \{0, 1\}$ for each $v \in V$ corresponds to a *dominating set* of G. A dominating set D is *minimal dominating* (i.e., no subset of D is dominating) if and only if each $v \in D$ dominates a vertex that is not dominated by $D - \{v\}$, that is, if and only if D is *irredundant*. Cockayne, Hedetniemi and Miller [6] introduced the concept of irredundance as precisely the property that makes a dominating set minimal dominating.

Ahmadi, Fricke, Schroeder, Hedetniemi and Laskar [1] use a property that makes a dominating broadcast minimal dominating, which was first mentioned in [9], to define broadcast irredundance, which we state here in Section 2.4. The *broadcast irredundance number* of G is defined as

$$\text{ir}_b(G) = \min\{\sigma(f) : f \text{ is a maximal irredundant broadcast of } G\}.$$

The definitions imply that $\text{ir}_b(G) \le \gamma_b(G)$ for all graphs G, and as our main result we prove that the ratio γ_b/ir_b is bounded:

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Theorem 1. *For any graph G, $\gamma_b(G) \leq \frac{5}{4}\operatorname{ir}_b(G)$.*

After defining our basic concepts in Section 2, we present some properties of irredundant broadcasts in Section 3, the most important of which is a necessary and sufficient condition for an irredundant broadcast to be maximal irredundant (Theorem 7). Theorem 1 is proved in Section 4. We briefly discuss upper broadcast domination and irredundance in Section 5, illustrating that the ratio IR_b / Γ_b is unbounded for general graphs, and conclude with a list of open problems and conjectures in Section 6.

2. Definitions

This section contains more definitions concerning dominating broadcasts, neighbourhoods and boundaries of broadcasting vertices, minimal dominating broadcasts and, finally, irredundant broadcasts. For undefined concepts we refer the reader to [4].

2.1. Dominating broadcasts

Consider a broadcast f on a connected graph $G = (V, E)$. Define $V_f^+ = \{v \in V : f(v) > 0\}$ and partition V_f^+ into the two sets $V_f^1 = \{v \in V : f(v) = 1\}$ and $V_f^{++} = V_f^+ - V_f^1$. A vertex u *hears* the broadcast f from some vertex $v \in V_f^+$, and v *f-dominates* u, if the distance $d(u, v) \leq f(v)$. Denote the set of all vertices that do not hear f by U_f; thus f is a dominating broadcast if $U_f = \varnothing$. A dominating broadcast f of G such that $\sigma(f) = \gamma_b(G)$ is called a *γ_b-broadcast*. A *radial broadcast* on G is a broadcast f such that $f(u) = \mathrm{rad}(G)$ for a central vertex u, and $f(v) = 0$ if $v \in V(G) - \{u\}$; clearly, a radial broadcast is dominating. Broadcast domination was introduced by Erwin [9], who also gave the trivial upper bound

$$\gamma_b(G) \leq \{\gamma(G), \mathrm{rad}(G)\}$$

for any graph G. Graphs for which $\gamma_b(G) = \mathrm{rad}(G)$ are called *radial graphs*. Radial trees are characterized in [11,12], where a formula for the broadcast number $\gamma_b(T)$ of a tree T, as well as a simple algorithm to determine $\gamma_b(T)$, can also be found. Another algorithm to determine $\gamma_b(T)$ is given in [7].

If f and g are broadcasts on G such that $g(v) \leq f(v)$ for each $v \in V$, we write $g \leq f$. If in addition $g(v) < f(v)$ for at least one $v \in V$, we write $g < f$. Also, $g \geq f$ ($g > f$, respectively) if $f \leq g$ ($f < g$, respectively). A dominating broadcast f on G is a *minimal dominating broadcast* if no broadcast $g < f$ is dominating. Clearly, a γ_b-broadcast is a minimal dominating broadcast, but the converse need not be true. The *upper broadcast number* of G, first defined in [9] and also studied in [1,8], is

$$\Gamma_b(G) = \max\{\sigma(f) : f \text{ is a minimal dominating broadcast of } G\},$$

and a dominating broadcast f of G such that $\sigma(f) = \Gamma_b(G)$ is called a *Γ_b-broadcast*. If f is a dominating broadcast such that $f(v) \in \{0, 1\}$ for each $v \in V$, then $\{v \in V : f(v) = 1\}$ is a dominating set of G; the smallest cardinality of a dominating set is the *domination number* $\gamma(G)$, and the largest cardinality of a minimal dominating set is the *upper domination number* $\Gamma(G)$. Again, Erwin [9] gave the trivial lower bound

$$\Gamma_b(G) \geq \max\{\Gamma(G), \mathrm{diam}(G)\}$$

for all graphs G.

Broadcast domination can be considered as an integer programming (IP) problem. Its fractional relaxation linear program (LP) has a dual linear program (DLP) whose IP formulation provides a lower bound for the broadcast number via the strong duality theorem of linear programming. The dual to the broadcast domination problem was referred to in [7] and studied explicitly by Teshima [14], Brewster, Mynhardt and Teshima [3], and Mynhardt and Teshima [13], who called it the *multipacking problem*. For a positive integer s, the *s-neighbourhood* $N_s[v]$ of $v \in V$ is the set of all vertices within distance s from v. A set M of vertices of G is called a *multipacking* if, for each $v \in V$ and each integer s such that $1 \leq s \leq e(v)$, the set $N_s[v]$ contains at most s vertices from M. The *multipacking number* $\mathrm{mp}(G)$ is the maximum cardinality of a multipacking of G. The duality of multipackings and broadcasts implies that $\gamma_b(G) \geq \mathrm{mp}(G)$ for any graph G. Hence the existence of a multipacking of cardinality m in a graph G with a dominating broadcast of cost m serves as a certificate that $\gamma_b(G) = m$.

2.2. Neighbourhoods and boundaries

For a set S of vertices of a graph G and $s \in S$, the *private neighbourhood* $\mathrm{PN}(s, S)$ of s with respect to S is the set of all vertices in the closed neighbourhood of s that are not in the closed neighbourhood of any other vertex in S. If $u \in \mathrm{PN}(s, S) - S$, then u is an *external private neighbour* of s. If $u \in \mathrm{PN}(s, S) \cap S$, then $u = s$, s is isolated in $G[S]$ and s is said to be an *S-self-private neighbour*. For a broadcast f on G and $v \in V_f^+$, define the

- *f-neighbourhood* of v as $N_f[v] = \{u \in V(G) : d(u, v) \leq f(v)\} = N_{f(v)}[v]$
- *f-boundary* of v as $B_f(v) = \{u \in V(G) : d(u, v) = f(v)\}$
- *f-private neighbourhood* of v as $\mathrm{PN}_f(v) = \{u \in N_f[v] : u \notin N_f[w] \text{ for all } w \in V^+ - \{v\}\}$
- *f-private boundary* of $v$as $\mathrm{PB}_f(v) = \{u \in N_f[v] : u \text{ is not dominated by } (f - \{(v, f(v))\}) \cup \{(v, f(v) - 1)\}\}$
- *external f-private boundary* of v as $\mathrm{EPB}_f(v) = \mathrm{PB}_f(v) - \{v\}$.