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Shortcutting directed and undirected networks with a degree constraint

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ABSTRACT

Shortcutting is the operation of adding edges to a network with the intent to decrease its diameter. We are interested in shortcutting networks while keeping degree increases in the network bounded, a problem first posed by Chung and Garey. Improving on a result of Bokhari and Raza we show that, for any $\delta \geq 1$, every undirected graph G can be shortcut in linear time to a diameter of at most $O(\log_{1+\delta} n)$ by adding no more than $O(n/\log_{1+\delta} n)$ edges such that degree increases remain bounded by δ. The result extends an estimate due to Alon et al. for the unconstrained case. Degree increases can be limited to 1 at only a small extra cost. For strongly connected, bounded-degree directed graphs Flaxman and Frieze proved that, if ϵn random arcs are added, then the resulting graph has diameter $O(\ln n)$ with high probability. We prove that $O(n/\ln n)$ edges suffice to shortcut any strongly connected directed graph to a graph with diameter less than $O(\ln n)$ while keeping the degree increases bounded by $O(1)$ per node. The result is proved in a stronger, parameterized form. For general directed graphs with stability number α, we show that all distances can be shortcut to $O(\alpha\lceil\ln \frac{n}{\alpha}\rceil)$ by adding only $\frac{4n}{\ln n/\alpha} + \alpha\phi$ edges while keeping degree increases bounded by at most $O(1)$ per node, where ϕ is equal to the so-called feedback-dimension of the graph. Finally, we prove bounds for various special classes of graphs, including graphs with Hamiltonian cycles or paths. Shortcutting with a degree constraint is proved to be strongly NP-complete and $W[2]$-hard, implying that the problem is neither likely to be fixed-parameter tractable nor efficiently approximable unless $FPT = W[2]$.

1. Introduction

Shortcutting is the operation of adding links (lines, edges) to a network with the intent to decrease its diameter. Shortcutting networks increases their transmission capacity and decreases network delay. Adding links to nodes in order to reduce a network's diameter is not free of charge, however. In many instances, the number of links that can be added to a node is limited due to physical or even economical constraints. Hence, in reality one may be able to add only a limited number of extra links per network node. It is this type of constraint that we are interested in.

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Specifically, we are interested in δ-*shortcutting* arbitrary networks G. In this problem we wish to shortcut a network subject to the constraint that the number of edges added per node is bounded by a fixed integer value $\delta \geq 1$. We require also that *added edges only link nodes that are connected in the transitive closure of* G, to remain faithful to the structure of the network. We model networks as finite graphs and phrase the shortcutting problem accordingly. We consider the following general question, for both undirected and directed n-node graphs G:

> *what reductions in existing node-to-node distances or diameter are achievable by shortcutting* G, *and how many extra edges are needed for it, when degree increases must remain limited by at most a small amount at every node.*

The study of shortcutting graphs *without* constraints seems to have been initiated by Chung and Garey [16]. In the undirected case, many studies show how to achieve small, even constant diameters with only a linear number of additional edges, both for special graph classes and in general (see e.g. [1,7,9,44]). The problem of determining whether some number of edges suffices to achieve a non-trivial reduction in diameter has been studied in many papers, leading e.g. to tight bounds, NP-hardness and even W[2]-hardness results [16,24,38]. Also the complexity of approximating the number of edges needed to achieve a certain diameter decrease has been studied [6,17,34]. Integer LP-techniques have been applied to it for achieving shortest paths between nodes in the expected case on certain classes of networks [10]. For the directed case, Thorup [42,43] showed that all m-edge planar digraphs can be shortcut to a poly-logarithmic diameter by the addition of at most m extra edges, but Hesse [27] showed that this fact does not hold for digraphs in general. Finally, a shortcutting of a graph may be viewed as a special case of a so-called *transitive-closure spanner* of the graph, although the latter has a different connotation (see e.g. [36]).

The shortcutting problem as we study it here, i.e. *with* degree constraint, seems to have received very little attention before. Chung and Garey [16] suggested to constrain the maximum degree of nodes in the problem, but few results seem to have been obtained for it. The only earlier study of δ-shortcutting seems to be due to Bokhari and Raza [8], who considered the problem for undirected graphs for the interesting case $\delta = 1$. In this paper we study the δ-shortcutting problem in general and aim at sharp bounds on the reductions in diameter or inter-node distances that can be achieved. In the sequel, when we speak of shortcutting we will always mean δ-shortcutting for some $\delta \geq 1$.

1.1. Results

We consider the shortcutting problem for the undirected and directed cases separately. We first review our results for *undirected* graphs. One fact can be noted straight off: a graph with maximum degree $\Delta \geq 3$ can have a diameter of $\log_{\Delta-1} n$ at best, based on the Moore bound (see Section 2). Hence, when a graph is δ-shortcut, one can hope for a diameter of about $\log_{1+\delta} n$ at best (taking $\Delta \geq 2 + \delta$ in the above bound).

For $\delta = 1$, the first results for the δ-shortcutting problem were obtained by Bokhari and Raza [8]. They showed that any connected undirected graph can be 1-shortcut to a diameter D with $D = O(\log_2 n)$, by adding at most n edges. They also showed that the edges needed for the shortcutting can be determined by an $O(n^2)$ algorithm. In Section 3 we improve on this, by giving an algorithm that 1-shortcuts a graph to a diameter D with $D = O(\log_2 n)$ by adding only $O(\frac{n}{\log_2 n})$ edges, by means of an $O(n)$ algorithm.

The improved bound is an instance of a more general result. Note that Alon et al. [1] (see also [35]) already showed that *without* degree constraints, any connected undirected graph G can be reduced to diameter D by adding at most $\frac{n}{\lfloor D/2 \rfloor}$ edges. We prove that any connected graph G can be shortcut in linear time to a diameter $O(D)$ by adding at most $\frac{n}{\lfloor D/2 \rfloor}$ edges while keeping degree increases smaller than $n^{\frac{2}{D}}$ (provided $D \geq 4$). Reformulating this for the shortcutting problem, the result states that for any integer $\delta \geq 1$, any connected undirected graph can be δ-shortcut in linear time to a diameter $O(\log_{1+\delta} n)$ by adding at most $O(\frac{n}{\log_{1+\delta} n})$ extra edges. The degree increases can be limited to 1 at the expense of an extra factor δ in diameter but saving a factor δ on the number of extra edges. As Alon et al. [1] proved their result to be worst-case optimal, so is our degree-constrained extension of it. We show this in Section 3.

Next we consider the shortcutting problem for *directed* graphs. Before turning to the general case, we prove several key results for δ-shortcutting rooted directed paths and δ-compressing rooted directed trees in Sections 4 and 5. These results are used in Section 6, e.g. to obtain useful bounds for 1-shortcutting DAGs and rooted directed trees depending on parameters like the width of the DAG or the height of the tree, respectively. As another step towards the general case, we consider the problem of shortcutting strongly connected digraphs.

For strongly connected digraphs, we note that Flaxman and Frieze [21] proved for the bounded-degree case that, if ϵn random arcs are added to the graph, then the resulting graph has diameter $O(\ln n)$ with high probability. We show that *any* strongly connected n-node digraph can be shortcut to a diameter of $O(\ln n)$, by adding $O(\frac{n}{\ln n})$ arcs and keeping degree increases bounded by 2. The bounds follow from a more general result that can be tuned in various further ways.

For general directed graphs we prove several bounds in Sections 6 and 7. In particular, in Section 7 we show that all distances in an n-node directed graph can be shortcut to $O(\alpha(G) \cdot \lceil \ln \frac{n}{\alpha(G)} \rceil)$, again by the addition of at most a sublinear number of arcs and keeping degree increases bounded by 2. Here $\alpha(G)$ is the *stability number* of G. The result involves an interesting application of the Gallai–Milgram theorem. As a corollary we show that every tournament can be 2-shortcut to a diameter $O(\ln n)$, by adding at most a linear number of arcs. To estimate the number of shortcut arcs needed in the results