Discrete Applied Mathematics 220 (2017) 161–169

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Embedding various cycles with prescribed paths into k-ary n-cubes[☆]

Yuxing Yang[a,*], Jing Li[b], Shiying Wang[a]

[a] School of Mathematics and Information Sciences, Henan Normal University, Xinxiang, Henan 453007, PR China

[b] School of Applied Science, Taiyuan University of Science and Technology, Taiyuan, Shanxi 030024, PR China

ARTICLE INFO

Article history:
Received 12 April 2016
Received in revised form 9 September 2016
Accepted 8 December 2016
Available online 5 January 2017

Keywords:
Multiprocessor systems
Interconnection networks
k-ary n-cubes
Cycles

ABSTRACT

The k-ary n-cube has been one of the most popular interconnection networks for large-scale multi-processor systems and data centers. In this study, we investigate the problem of embedding cycles of various lengths passing through prescribed paths in the k-ary n-cube. For $n \geq 2$ and $k \geq 5$ with k odd, we prove that every path with length $h\,(1 \leq h \leq 2n-1)$ in the k-ary n-cube lies on cycles of every length from $h+(k-1)(n-1)/2+k$ to k^n inclusive.

1. Introduction

In parallel and distributed systems, processors are connected based on a given interconnection network and the interconnection network plays a crucial role in hardware costs, communication performance and potentialities for efficient algorithms [19]. The k-ary n-cube, denoted by Q_n^k, as one of the most attractive interconnection networks, has drawn considerable research attention for its desirable properties, such as ease of implementation, low-latency and high-bandwidth inter-processor communication [4,9,12,13]. Many parallel systems, such as the Mosaic [27], the iWarp [26], the J-machine [25], the Cray T3D [17], the Cray T3E [2], the IBM Blue Gene [1] and a data center [8], have been built with the k-ary n-cube forming underlying topology.

Among the existing interconnection networks, the cycle (also called ring) network and path are two of the most attractive owing to their simple structures and low degrees, which are helpful to design simple routing algorithms with low communication costs. Thus, the problem of embedding cycles into famous interconnection networks has been researched in depth (see, for example, [6,18,23] and the references therein).

In real systems, failures of processors and/or links are inevitable. Therefore, when embedding cycles and paths in their interconnection networks, these faulty components are unavailable and need to be avoided. On the other hand, maybe some processors and/or links have better performance than others in real systems. When embedding cycles and/or paths in their interconnection networks, one would prefer to choose the stronger nodes and/or edges. In this scenario, Caha and Koubek [5] introduced the "prescribed embedding" problem. They investigated the hamiltonian cycle and path embedding problem in 2-ary n-cubes with prescribed edges, and observed that the subgraph induced by any proper subset of edges of a hamiltonian

☆ This work is supported by the Joint Fund of NSFC and Henan Province (U1304601), NSFC (61370001, 61303020) and Scientific Research Project Fund of Henan Provincial Education Department (14B520004).

* Corresponding author.
E-mail addresses: yxyangcn@163.com, yyx@htu.edu.cn (Y. Yang).

http://dx.doi.org/10.1016/j.dam.2016.12.006
0166-218X/

cycle necessarily is a linear forest. Since then, the problem of embedding cycles passing through a linear forest in the k-ary n-cube has been studied in several literatures (see, for example, [5,7,10,11,21,31,34,33,36–38]). Among the above, Dvořák [10] proved that the 2-ary n-cube contains a hamiltonian cycle passing through a linear forest with at most $2n - 3$ edges. Wang et al. [31] investigated the hamiltonian cycle embedding problem in 2-ary n-cubes with a prescribed linear forest and faulty edges. Wang et al. [33] showed that a k-ary n-cube admits a Hamiltonian cycle passing through a prescribed linear forest with at most $2n - 1$ edges.

It is well known that an interconnection network is usually represented by an undirected simple graph $G = (V(G), E(G))$. A graph G is said to be *m-pancyclic* if it contains cycles of every length from m to $|V(G)|$ inclusive, and *bipancyclic* if it contains cycles of every even length from 4 to $|V(G)|$. Furthermore, an m-pancyclic graph G is said to be *pancyclic* if m is the length of the shortest cycle of G, and *p-path q-pancyclic* if every path with length h, $1 \le h \le p$ lies on cycles of every length from $h + q$ to $|V(G)|$ inclusive; a bipancyclic graph is said to be *p-path bipancyclic* if every path with length h, $1 \le h \le p$ lies on cycles of every even length from $\max\{2h, g(G)\}$ to $|V(G)|$ inclusive, where $g(G)$ is the length of the girth of G. In particular, if $p = 1$, a p-path q-pancyclic (resp. p-path bipancyclic) graph is said to be $(q + 1)$-edge-pancyclic (resp. edge-bipancyclic). The above properties are important measurements to determine if a network topology is suitable for a real application where mapping various cycles with prescribed links into the network topology is required. In recent years, the pancyclicity, bipancyclicity, path pancyclicity and path bipancyclicity of the k-ary n-cube have been researched in many literatures (see, for example, [14–16,20,22,24,28,29,32,35]). For the 2-ary n-cube, Tsai and Jiang [30] showed that the 2-ary n-cube is $(2n-4)$-path-bipancyclic; Chen [7] investigated the edge-bipancyclic of 2-ary n-cubes with faulty edges. Recently, Tsai [29] studied the path bipancyclicity of the 2-ary n-cube and so obtained a more general result. For the 3-ary n-cube, Li et al. [21] proved that the 3-ary n-cube with $n \ge 2$ is $(2n - 1)$-path $(n + 1)$-pancyclic. To the best of our knowledge, the problem of embedding cycles passing through prescribed paths in the k-ary n-cube with general k remains yet to be investigated. The k-ary n-cube with general k has better application prospect than its special cases (i.e., the 2-ary n-cube and the 3-ary n-cube). To continue Caha's work in the domain of "prescribed embedding problem" and to generalize the results of Li et al. [21], we will study the path pancyclicity of the k-ary n-cube with $n \ge 2$ and odd $k \ge 5$ and prove that the k-ary n-cube with $n \ge 2$ and odd $k \ge 5$ is $(2n - 1)$-path $((k - 1)(n - 1)/2 + k)$-pancyclic.

The remainder of this paper is organized as follows. In Section 2, we introduce some basic definitions, notation and results. In Section 3, we study the path pancyclicity of k-ary 2-cubes. In Section 4, we prove our main result. Concluding remarks are covered in Section 5.

2. Definitions and preliminaries

Throughout this paper, we mostly follow [3] for the graph-theoretical terminology and notation not defined here. Let G be an undirected simple graph. Suppose that $V' \subseteq V(G)$. The subgraph of G *induced by* V', denoted by $\langle V' \rangle$, is the subgraph of G whose vertex set is V' and whose edge set consists of all edges of G which have both end vertices in V'. The induced subgraph $\langle V(G) \setminus V' \rangle$ is denoted by $G - V'$. The distance of vertices u and v in G is denoted by $dist_G(u, v)$ and the degree of v in G is denoted by $d_G(v)$.

A *u–v path*, denoted by $P = \langle x_0, x_1, \ldots, x_t \rangle$, is a sequence of different vertices such that two consecutive vertices are adjacent. We say that a cycle C *passes through* a path P if $E(P) \subseteq E(C)$. A graph G is said to be *p-panconnected* if for any two distinct vertices $u, v \in V(G)$, there is a u–v path of every length from p to $|V(G)| - 1$ inclusive. In particular, if $p = dist_G(u, v)$, then G is said to be *panconnected*.

The *k-ary n-cube* $Q_n^k (k \ge 2, n \ge 1)$ is a graph consisting of k^n vertices, each of which has the form $u = \delta_1\delta_2 \ldots \delta_n$, where $0 \le \delta_i \le k - 1$ for all $1 \le i \le n$. Two vertices $u = \delta_1\delta_2 \ldots \delta_n$ and $v = \sigma_1\sigma_2 \ldots \sigma_n$ are adjacent if and only if there exists an integer $j \in \{1, 2, \ldots, n\}$ such that $\delta_j = \sigma_j \pm 1 \pmod k$ and $\delta_l = \sigma_l$, for every $l \in \{1, 2, \ldots, n\} \setminus \{j\}$. Such edge (u, v) is an *edge in dimension j*. It is not difficult to see that each vertex of Q_n^k has $2n$ neighbors in Q_n^k when $k \ge 3$.

For any $d \in \{1, 2, \ldots, n\}$, we can *partition* Q_n^k *over dimension* d, by deleting all the edges in dimension d, into k disjoint copies of Q_{n-1}^k, namely, $Q_{n-1}^k[0], Q_{n-1}^k[1], \ldots, Q_{n-1}^k[k - 1]$ (abbreviated as $Q[0], Q[1], \ldots, Q[k - 1]$, if there are no ambiguities). In the remainder of this paper, for $i \in \{0, 1, \ldots, k - 1\}$, we prefer to abbreviate $E(Q[i])$ and $V(Q[i])$ as E_i and V_i, respectively. Let $u^i = \delta_1\delta_2 \ldots \delta_{d-1} i \delta_{d+1} \ldots \delta_n$ be an arbitrary vertex of $Q[i]$. Then for $j \in \{0, 1, \ldots, k - 1\} \setminus \{i\}$, the vertex $u^j = \delta_1\delta_2 \ldots \delta_{d-1} j \delta_{d+1} \ldots \delta_n$ is called the *corresponding vertex* of u^i in $Q[j]$.

The following two results are useful in later discussion.

In [14,15], Hsieh et al. studied the problem of path embeddings in the k-ary n-cube and they proved the following.

Lemma 1 (*See [14]*)**.** *Let $n \ge 2$ be an integer and $k \ge 3$ be an odd integer. Then Q_n^k is $n(k - 1)/2$-panconnected and k-edge-pancyclic.*

In [24], Lin et al. investigate the panconnectivity of the k-ary n-cube (odd $k \ge 3$) with some faulty vertices and/or edges and they obtained the following.

Lemma 2 (*See [24]*)**.** *Let $n \ge 2$ be an integer and $k \ge 3$ be an odd integer. Then $Q_n^k - F$ is $(n(k - 1) - 1)$-panconnected for any set F of faulty vertices and/or edges with $|F| \le 2n - 3$.*