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Discounted tree solutions

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ABSTRACT

This article introduces a discount parameter and a weight function in the classical model of cooperative games with restrictions on cooperation, represented by a tree. We provide axiomatic characterizations of solutions that are inspired by the hierarchical outcomes.

1. Introduction

A cooperative game with transferable utility, or simply a TU-game, consists of a finite set of agents N and for every coalition of agents a worth representing the total payoff that the coalition can obtain by cooperating. Myerson [14] introduced TU-games with restricted cooperation possibilities (graph TU-games henceforth) modeled by an undirected communication graph. The set of nodes is regarded as the set of agents and the (undirected) edges represent the bilateral communication possibilities between these agents. Myerson defined a (graph-)restricted TU-game based on the idea that only connected coalitions are likely to form. In this article, we enrich Myerson's model by two new elements:

- A *discount parameter* $\delta \in [0, 1]$, which changes the power or reservation worth of the coalitions.
- A *weight function* p, which assigns a weight $p(i) \in \mathbb{R}$ to each agent $i \in N$ and reflects the importance or social power of i.

Joosten [11] incorporated a discount parameter in TU-games and introduced the discounted Shapley values for TU-games. This class of solutions contains the Shapley value [16] and the equal division solution as extreme cases and thus belongs to the popular class of solutions which create space for solidarity (see Casajus and Huettner [6] for a recent study of another class of such values).

Contrary to Joosten [11], we incorporate the discount rate in combination with a tree structure.[1] A first possible motivation for this model is as follows. In many solutions for TU-games, the payoffs distributed to the agents are sequentially

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URL: https://sites.google.com/site/bealpage/ (S. Béal).

[1] The assumption that the cycle-free graph is a tree is only made for the sake of exposition.

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obtained through a process of coalition formation, in which the agents successively come into play and join the current coalition until the grand coalition is formed. Following in the footsteps of Demange [8], we assume that agents enter into coalitions according to a partial order represented by a rooted tree. Agents who are located farthest from the root enter into empty coalitions to form singleton coalitions, followed by their predecessors, and so on up to the root. When an agent comes into play, he or she joins several coalitions formed at the previous stage of the process. Precisely, an agent joins the coalition which contains all her or his subordinates in the rooted tree. In a rooted tree, agent's payoff is equal to her or his contribution to these coalitions, taking into account the fact that the worth of each coalition is discounted, through the parameter δ, according to its distance from the root. These payoff vectors, one for each possible rooted tree, are called discounted hierarchical outcomes in reference to the (undiscounted) hierarchical outcomes introduced by Demange [8] in tree TU-games. In addition to the worth of the coalitions and the communication structure, exogenous strength or power of the agents may have an impact on the total worth to be distributed and also on the way to distribute it. Indeed, it is thus quite common that agents exercise their power (financial or not) to influence the outcome of an interaction. The introduction of a weight function reflects this social power. Section 3 offers a strategic or non cooperative interpretation of the discounted hierarchical outcomes and their weighted sum.

This being said, this article belongs to the literature that deals with the axiomatic foundation of solution concepts for cooperative games. Section 4 contains the main characterization result on the model enriched by both the discount parameter and the weight function. Proposition 1 is an axiomatic characterization of the weighted sum of the discounted hierarchical outcomes, such that the coordinates associated with each hierarchical outcome coincide with the weight of the corresponding rooted agent. We invoke three axioms: node-weighted efficiency, invariance with respect to cone amalgamation and generalized standardness. Node-weighted efficiency captures the above mentioned idea that agents can exercise their power to influence the total worth to be distributed. This axiom indicates that the agents altogether achieve a total output that depends on both the worth of the grand coalition (the production resulting from their cooperation of all agents) and their power in the situation. This power can stimulate or restrain the worth of the grand coalition in proportion to the cumulated weights. Precisely, node-weighted efficiency says that the sum of the agents' payoff equals the above total output. Thus, beyond the classical communicational effects that are often taken into account in the literature, we also incorporate an external effect, which can be positive or negative depending on the sum of the agents' weight. Invariance with respect to cone amalgamation implements an amalgamation principle as follows. If the agents in a proper cone[2] are amalgamated and act as a single entity, then the payoff of any other agent is not affected. Our amalgamated TU-game is in line with those initiated by Lehrer [13]. Generalized standardness extends in a natural way the well-known axiom of standardness (see Hart and Mas-Colell [9]) to situations with both a discount parameter and a weight function. It turns out that these three axioms characterize the solution defined as the linear combination of discounted hierarchical outcomes such that the coordinates associated with each hierarchical outcome coincide with the weight of the corresponding root (Proposition 1).

In Section 5, we study the model enriched by only the discount parameter. Proposition 2 is an axiomatic characterization of the set of all linear combinations of the discounted hierarchical outcomes, by means of linearity, proper cone efficiency and the axiom of δ-reducing agent. Linearity and proper cone efficiency are already used in Béal et al. [4] where the linear combinations of the (undiscounted) hierarchical outcomes are characterized. Proper cone efficiency only requires efficiency for standard characteristic functions associated with the proper cones of a tree. The axiom of δ-reducing agent is a variation of the axiom of null agent in Béal et al. [4] that accounts for the discount parameter in a similar way as in van den Brink and Funaki [20]. Furthermore, by definition, for each tree and each discount rate, the set of discounted tree solutions forms a linear space. Proposition 2 also establishes that the set of hierarchical outcomes forms a basis for this space, which means that the dimension of this space is equal to the size of the agent set. Two particular cases similar to those arising with the discounted Shapley values are obtained: the so-called marginalist tree solutions (Theorem 1 in Béal et al. [4]) if $\delta = 1$, and the equal division solution if $\delta = 0$. As a by-product, we improve Theorem 1 in Béal et al. [4] by showing that the axiom of cone equivalence is unnecessary to obtain this result (Lemma 1). Adding the mild axiom of communication ability to the axioms in Proposition 2 and invoking efficiency instead of proper cone efficiency yield a characterization of the average discounted tree solution (see Proposition 3), which generalizes the average tree solution for cycle-free graph TU-games introduced by Herings et al. [10]. Communication ability assigns identical payoffs to all agents if all coalitions except the grand coalition enjoy a zero worth. Other generalizations of the average tree solution are due to Béal et al. [3] for multichoice graph TU-games, to Khmelnitskaya and Talman [12] for cycle-free directed graph TU-games and to van den Brink et al. [21] for graph TU-games with a permission tree. Section 5 contains also a comparable characterization of the solution studied in Section 2 on the model enriched by both the discount parameter and the weight function. Because this solution satisfies linearity, proper cone efficiency, the δ-reducing axiom, it is possible to characterize it by node-weighted efficiency, which implies proper cone efficiency, the δ-reducing axiom, a version of linearity that takes into account the fact that the solution is additive on p, and a modified version of communication ability.

Finally, Section 2 is devoted to the definitions and notations, and Section 6 concludes.

[2] A proper cone corresponds to a subtree associated with one of the two components that are obtained after the deletion of an edge. See, Section 2 for a formal definition.