Discrete Applied Mathematics 219 (2017) 32–39

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Majority dynamics with one nonconformist

John Haslegrave [a,*], Chris Cannings [b]

[a] University of Warwick, Coventry, UK
[b] University of Sheffield, Sheffield, UK

ARTICLE INFO

Article history:
Received 17 June 2016
Received in revised form 24 November 2016
Accepted 8 December 2016
Available online 2 January 2017

Keywords:
Majority dynamics
Threshold automata
Voter model
Social learning
Periodicity

ABSTRACT

We consider a system in which a group of agents represented by the vertices of a graph synchronously update their opinion based on that of their neighbours. If each agent adopts a positive opinion if and only if that opinion is sufficiently popular among his neighbours, the system will eventually settle into a fixed state or alternate between two states. If one agent acts in a different way, other periods may arise. We show that only a small number of periods may arise if natural restrictions are placed either on the neighbourhood structure or on the way in which the nonconforming agent may act; without either of these restrictions any period is possible.

1. Introduction

We consider a general setting in which a number of agents with a system of neighbourhood relationships have binary opinions which they update synchronously based on their neighbours' opinions. Neighbourhood is an arbitrary symmetric relation, and we represent the agents as vertices of a graph with edges, and, if necessary, loops, corresponding to the neighbourhood relation. Perhaps the most natural model for such updating of opinion is for each agent to adopt the more popular opinion among its neighbours (majority dynamics). A more general model along the same lines is to allow each agent to be inclined against a particular opinion, only adopting that opinion if sufficiently many neighbours (not just a simple majority) do. Different agents can be inclined towards different opinions or to different degrees. Such a system forms a threshold network; threshold networks were introduced by McCulloch and Pitts [9] to model activation of neurons. They also arise naturally as myopic best response strategies in networks of agents playing a coordination game (see e.g. [2,14]).

Majority dynamics and the more general threshold networks have been much studied. A classical result is the period-2 property. Since any finite threshold network has only a finite number of states and the progression from state to state is deterministic and memoryless, periodic behaviour must eventually arise from any possible starting state. What lengths of period are possible? It is not obvious that there is any constant bound on the period, but in fact the only possible periods are 1 and 2. This was proved independently by Goles and Olivos [6] (see also [5]) and Poljak and Sûra [12]. Poljak and Turzík [13] gave good bounds on the time until periodic behaviour begins.

If the network is infinite then the system does not necessarily reach a periodic state. Furthermore, even if it does, any period can occur. Moran [10] showed that with the additional conditions of bounded neighbourhoods and subexponential growth, both of which are necessary, again only periods 1 and 2 are possible. Ginosar and Holzman [4] show that under suitable conditions on an infinite graph a local period-2 property holds, in that each agent will eventually have a constant

* Corresponding author.
E-mail addresses: j.haslegrave@cantab.net (J. Haslegrave), c.cannings@sheffield.ac.uk (C. Cannings).

http://dx.doi.org/10.1016/j.dam.2016.12.004
0166-218X/

or alternating opinion (though the system as a whole may never become periodic since the times at which agents settle into these patterns could be unbounded).

If the updates are asynchronous, it is easy to see that on any finite threshold network the state is ultimately constant. Tamuz and Tessler [16] showed that this again holds locally on an infinite graph under suitable conditions. This contrasts with the closely related zero-temperature stochastic Ising model, which updates as asynchronous majority dynamics except that ties are broken randomly; Nanda, Newman and Stein [11] showed that under the latter model on $\mathbb{Z}^2$, almost surely every vertex changes its opinion infinitely often.

Other facets of majority dynamics have been studied, such as the question of whether a bias in the initial opinions tends to be preserved by this process (Tamuz and Tessler, [16]), and the threshold of initial bias which results in consensus on infinite trees (Kanoria and Montanari, [8]). Probabilistic versions of majority dynamics have been studied on highly-structured graphs. A model where agents make synchronous updates to the majority opinion among their neighbours, but occasionally make errors, dates back at least to work by Gray from the 1980s [7], but a similar model was considered significantly earlier by Spitzer [15]. Most studies on this model are merely computer simulations, but the few rigorous results include Gray's proof that the 1-dimensional version does not have a phase transition [7] and, more recently, the result of Balister, Bollobás, Johnson and Walters [1] that if the probability of error is small then the 2-dimensional torus spends almost all its time in a consensus state.

The opposite notion to majority dynamics, where each agent adopts the minority opinion of its neighbourhood, also arises naturally from the myopic best response strategy for a congestion game [14]. We may similarly generalise this to an anti-threshold network, where each agent adopts an opinion if it is sufficiently unpopular in the neighbourhood. The period-2 property for finite anti-threshold networks follows immediately from the result on threshold networks.

Cannings [3] considered various situations on simple graphs in which there were both majority and minority agents present, showing that cycles of various lengths could occur. He analysed particularly the case of a complete graph, proving that only cycles of length 1, 2 and 4 are possible, with length 4 only occurring in the special case of having equal numbers of majority and minority agents. A further class of cubic graphs was considered and possible cycle lengths for various numbers of minority agents obtained by direct simulation. A striking feature of these data is that when only one agent makes minority updates while the others make majority updates, only periods 1, 2 and 4 appear. However, this is not true for all graphs (or even all cubic graphs, e.g. Fig. 1(b)).

A consequence of the way the cubic graphs considered in [3] are constructed is that they will have no triangles. We show that it is true for all triangle-free simple graphs that only periods 1, 2 and 4 arise for majority dynamics with one additional agent following a different protocol. This result applies in the much more general setting where the nonconforming agent updates his opinion as any function of its neighbours' opinions, not necessarily choosing the minority opinion, and also if triangles are permitted so long as the nonconformist is not part of any triangle. If loops are permitted then there are more possibilities, but we prove that only a few different periods can arise. We also show that if the nonconforming agent does update to the minority opinion of his neighbours, then again only a few different periods can arise, with no restriction on triangles.

We will prove all our results for the general threshold situation, but they could equivalently be re-stated in terms of majority updates. It is easy to see that an agent updating according to an arbitrary threshold may be simulated by a suitable bundle of majority agents, and so any dynamics arising from arbitrary threshold networks with one nonconformist can also arise from majority dynamics with one nonconformist on a larger graph. This larger graph can also easily be chosen in accordance with the various restrictions on graphs that we consider.

Formally, we fix a finite graph G, which may have loops but not multiple edges, on vertex set $\{v_1, \ldots, v_n\}$. For each i, write N_i for the neighbourhood of v_i (including v_i if there is a loop there). The graph is initialised by giving each vertex one of two opinions, which we represent as $\{+1, -1\}$, at time 0, and all vertices simultaneously update their opinions at each time step. Write U_t for the set of vertices having opinion $+1$ at time t. Each vertex v_i has an update rule which depends only on the state of N_i at the previous time step, i.e. for each i there is a set system $\mathcal{S}_i \subseteq \mathcal{P}N_i$ such that $v_i \in U_{t+1}$ if and only if $N_i \cap U_t \in \mathcal{S}_i$. We say that v_i has a *threshold rule* with threshold r_i if $\mathcal{S}_i = \{A \subseteq N_i : |A| \geqslant r_i\}$ for some r_i, and an *anti-threshold rule* if $\mathcal{S}_i = \{A \subseteq N_i : |A| < r_i\}$. We will always assume that every vertex except v_1 has a threshold rule, with v_i having threshold r_i for $i > 1$. Write U_t^* for $U_t \setminus \{v_1\}$.

2. A Lyapunov operator

Proofs of the period-2 property for threshold networks (see [6,12,5,2]) proceed by defining a suitable Lyapunov operator, proving that it is bounded, integer-valued and non-decreasing, so must be ultimately constant, and showing that if at any step the value does not change then the state is identical to the previous state but one. In this section we give a modified Lyapunov operator for the situation where v_1 has an arbitrary rule, and show that provided every other vertex has a threshold rule this is still bounded and non-decreasing, and must be an integer multiple of 1/2, so is ultimately constant. The analysis of what can happen once this operator has reached its final value is much more complicated than for pure threshold networks, and we carry out this analysis separately for triangle-free graphs in Section 3 and for general graphs with v_1 having an anti-threshold rule in Section 4.

Theorem 1. *For t sufficiently large, if $v_i \in U_{t-1}^* \triangle U_{t+1}^*$ then $v_i \in N_1$ and $|N_i \cap U_t^*| = r_i - 1$.*