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Misère Nim with multi-player

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ABSTRACT

Krawec (2012) introduced a method of analyzing multi-player impartial games, and derived a recursive function capable of determining which of the n players has a winning strategy. The present paper is devoted to the game "Misère N-pile Nim with n players", abbreviated by MiNim(N, n), assuming that the standard alliance matrix is adopted. The game values of MiNim(N, n) are completely determined for three cases: $n > N + 1$, $n = N + 1$ and $n = N$. The case $n < N$ is more complicated, we present the game values of MiNim(N, n) only for $n = 3$ and $N = 4$.

1. Introduction

Combinatorial game theory is a branch of mathematics devoted to studying the optimal strategy in perfect-information games where typically two players are involved. In a 2-person perfect information game two players alternate moves until one of them is unable to move at his turn. Among the games of this type are *Nim* [1,4,11,14,22], *End-Nim* [2,12], *Wythoff's game* [6,8–10,21], *a-Wythoff's game*, (s, t)-*Wythoff's game* [20,23], *Wythoff-like game* [13,24], etc. There are two conventions: in *normal play convention*, the player first unable to move is the loser (his opponent is the winner); in *misère play convention*, the player first unable to move is the winner (his opponent is the loser). The positions from which the previous player can win regardless of the opponent's moves are called *P-positions* and those from which the next player can win regardless of the opponent's moves are called *N-positions*. The theory of such games can be found in [3,7].

1.1. Two-player combinatorial game

Nim with two players is played with piles of counters. The two players take turns removing any positive integer of counters from any one pile. Under normal play convention, Bouton's analysis of Nim [4] showed that the P-positions are those for which nim-addition on the sizes of the piles is 0, and the N-positions are those for which nim-addition on the sizes of the piles is greater than 0. In the same paper, all P-positions of Nim were determined under misère play convention.

Of particular interest in 2-player combinatorial game theory is the Sprague–Grundy function [3] defined recursively by

$$\mathcal{G}(G) = \begin{cases} 0, & \text{if } G = \emptyset, \\ \text{mex}\{\mathcal{G}(G_i) \mid G_i \in Opt(G)\}, & \text{otherwise,} \end{cases}$$

where "mex" returns the minimal excluded non-negative integer, and $Opt(G)$ denotes the set of all games attainable by a legal move from G. In [3], the authors showed that under normal play convention, a game G is a P-position if and only if

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$\mathcal{G}(G) = 0$, or an N-position if and only if $\mathcal{G}(G) > 0$. Thus we may use the Sprague–Grundy function to determine whether or not the first or second player has a winning strategy. However, the Sprague–Grundy function does not apply to more than two players games.

1.2. Multi-player combinatorial game

During the last few years, the theory of 2-player perfect information games has been promoted to an advanced level. Naturally it is of interest to generalize as much as possible of the theory to n-player games. In 2-player perfect information games, one can always talk about what the outcome of the game *should* be, when each player plays it right i.e., when each player adopts an optimal strategy. But when there are more than two players, it may not make sense to talk about the same thing. For instance, it may so happen that one of the players can help any of the players to win, but anyhow, he himself has to lose. So the outcome of the game depends on how the group coalitions are formed among the players. In previous literatures, several directions were investigated: *multi-player without alliance*, *multi-player with two alliances* and *multi-player with alliance system*.

1.2.1. Multi-player without alliance

The game *n-player Nim without alliance* was introduced by Li [19]: The n players are $P_1, P_2, \ldots, P_n$, according to the initial order of turns. The players take turns moving counters from one pile of $(c_1, c_2, \ldots, c_p)$. The game is ended when any player is unable to move at his turn. Naturally under normal play convention, we define the loser to be the first player unable to move. If that player is P_m, say, we assign a different rank to each player, ranking from bottom to top in the order of $P_m, P_{m+1}, \ldots, P_n, P_1, P_2, \ldots, P_{m-1}$. In particular, the last player able to move is the *top* winner. Under these rules, the rank of any one player automatically determines the ranks for all. For this reason, it makes sense to say what the outcome of the game should be when each player adopts an optimal strategy toward his own highest possible rank.

Straffin [27] attempted to classify three-player games using somewhat restrictive assumptions regarding the behavior of each player. This work was also investigated by Loeb [25] by introducing the notion of a stable winning coalition (where a member of this coalition is guaranteed a winner).

Other work done by Propp [26] analyzed the required circumstances which allow one player to have a winning strategy against the combined forces of the others. Cincotti [5] gave an analysis of n-player partizan games.

1.2.2. Multi-player with two alliances

The game of *n-player one-pile bounded Nim with two alliances*, denoted by **OBN**, was investigated in [15,16]: given an integer $m \geq 1$ and a pile of counters, suppose that $n \geq 2$ players form two alliances and that each player is in exactly one alliance. Also assume that each player will support his alliance's interests. Each player is allowed to remove ℓ counters from the pile, where $\ell \in \{1, 2, \ldots, m\}$. Under misère play convention, the alliance which takes the last counter is the loser (the other alliance is the winner); under normal play convention, the alliance which takes the last counter is the winner (the other alliance is the loser).

A position is defined to be *an unsafe position of one alliance* if the game begins from this position and no matter what move this alliance makes, when the other alliance plays optimally, this alliance must lose. In misère play convention, Kelly [15,16] gave all unsafe positions of **OBN** for some special structures of two alliances.

More general structures of two alliances were investigated by Zhao and Liu [28], and all unsafe positions of **OBN** were determined for more general structures of two alliances. Moreover, the authors also pointed out that some conclusions given by Kelly are not correct, and presented a possible explanation for Kelly's inaccurate conclusions.

1.2.3. Multi-player with alliance system

Krawec ([17], 2012) assumed that every player has a fixed set of allegiances to all n players i.e., an *alliance system* may be defined arbitrarily before the start of a game. While the alliance system used is fixed for the duration of the game, Krawec provided a method of analyzing n-player impartial games, and derived a recursive function capable of determining which of the n players has a winning strategy.

Krawec ([18], 2015) developed a method of analyzing n-player impartial combinatorial games where $n-1$ players behave optimally while one of the players plays randomly i.e., one player chooses games at random without strategy.

1.3. Our games and results

Definition 1. We introduce a class of n-player impartial games, denoted by *MiNim(N, n)*: There are N piles of counters. The n players take turns in sequential unchanging order. Each player at his turn removes *any* positive integer of counters from *any* one pile. The first player who cannot make any legal move wins.

The present paper is devoted to MiNim(N, n) for all integers N and n, assuming that *Standard Alliance Matrix* is adopted. By applying the "game value function" introduced by Krawec in [17], the game values of MiNim(N, n) are completely determined for three cases:

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