ARTICLE IN PRESS
Discrete Applied Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

On the independence transversal total domination number of graphs

Abel Cabrera Martínez [a], José M. Sigarreta Almira [a], Ismael G. Yero [b,*]

[a] Facultad de Matemáticas, Universidad Autónoma de Guerrero, Carlos E. Adame 5, Col. La Garita, Acapulco, Guerrero, Mexico
[b] Departamento de Matemáticas, Escuela Politécnica Superior de Algeciras, Universidad de Cádiz, Av. Ramón Puyol s/n, 11202 Algeciras, Spain

ARTICLE INFO

Article history:
Received 9 July 2015
Received in revised form 16 October 2016
Accepted 29 October 2016
Available online xxxx

Keywords:
Independence transversal total domination number
Total domination number
Independence number

ABSTRACT

A total dominating set of a graph G having non empty intersection with all the independent sets of maximum cardinality in G is an independent transversal total dominating set. The minimum cardinality of any independent transversal total dominating set is denoted by $\gamma_{tt}(G)$. In this paper we introduce this concept and begin the study of its mathematical properties. Specifically, we prove that the complexity of the decision problem associated to the computation of the value of $\gamma_{tt}(G)$ is NP-complete, under the assumption that the independence number is known. Moreover, we present tight lower and upper bounds on $\gamma_{tt}(G)$ and give some realizability results in concordance with these bounds. For instance, we show that for any two positive integers a, b such that $2 \leq a \leq \frac{2b}{3}$ there is a graph G of order b such that $\gamma_{tt}(G) = a$.

1. Introduction

Throughout this work we only consider simple graphs G with a vertex set $V(G)$ and an edge set $E(G)$. That is, graphs that are finite, undirected, and without loops or multiple edges. Given a vertex v of G, we say that $N_G(v)$ represents the set of all neighbors of v in G and the degree of v is $\delta(v) = |N_G(v)|$. The minimum and maximum degrees of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. For short, we will often use $N(v)$ instead of $N_G(v)$.

A subset D of $V(G)$ is a *dominating set* in G if every vertex in $V(G) - D$ is adjacent to at least one vertex in D. The *domination number* of G is the smallest size of any dominating set in G and is denoted by $\gamma(G)$. A $\gamma(G)$-set is a dominating set of cardinality $\gamma(G)$. Moreover, the set D is a *total dominating set* in G if every vertex in $V(G)$ is adjacent to at least one vertex in D. The *total domination number* of G is the shortest size of any total dominating set in G and is denoted by $\gamma_t(G)$. A $\gamma_t(G)$-set is a total dominating set of cardinality $\gamma_t(G)$. For more information on domination and total domination see the book [8] and the survey [9].

Given a set C of objects and a collection of subsets of C, a transversal of the collection of subsets is given by a set of distinct representative objects of each one of the elements of the collection. Transversals of different kinds of vertex subsets of a graphs have been frequently studied in the last decades. Examples of transversals concerning the chromatic number or the independence number of a graph are commonly known. For instance, the article [4] was focused on the following problem. Consider a partition of the vertex set of a graph satisfying a bound (lower or upper) on the quantity of elements in each set of the partition. In this sense, is there a transversal of the partition being an independent set or a dominating set?

* Corresponding author.
E-mail address: ismael.gonzalez@uca.es (I.G. Yero).

http://dx.doi.org/10.1016/j.dam.2016.10.033

A few good results on this problem were presented in Fellows [4]. For example, we remark some possible applications to fault-tolerant data storage and the complexity analysis of the associated decision problems. The example mentioned above is not the only transversal-type parameter defined and studied until now. However, according to the amount of literature about this topic in each of its related variants, we restrict our references principally to those ones which are only citing papers that we really refer to in a non superficial way. More recently, some new style of transversal has been presented in Hamid [7]. That is the independence transversal domination number. According to its novelty, this parameter remains relatively unknown and just a few interesting results on it are published in Abdollahzadeh Ahangar et al. [1] and Hamid [7].

A set S of vertices is *independent* if the subgraph induced by S is edgeless. An independent set of maximum cardinality is a *maximum independent set* of G. The *independence number* of G is the cardinality of a maximum independent set of G and is denoted by $\beta(G)$. An independent set of cardinality $\beta(G)$ is called a *$\beta(G)$-set*.

A dominating set of G which intersects every independent set of maximum cardinality in G is called an *independent transversal dominating set*. The minimum cardinality of an independent transversal dominating set is called the *independence transversal domination number* of G (see Hamid [7]).

In the present article we introduce a new variant of transversals in graphs, which arise from the above mentioned independence transversal domination number and the total domination number in graphs. That is, a total dominating set of G which intersects every independent set of maximum cardinality in G is called an *independent transversal total dominating set* (or TTD set for short). The minimum cardinality of an ITTD set is called the *independence transversal total domination number* of G and is denoted by $\gamma_{tt}(G)$. An ITTD set of cardinality $\gamma_{tt}(G)$ is a *$\gamma_{tt}(G)$-set*.

As an example, we can observe the graph G_{10} of Fig. 1 where $\beta(G_{10}) = 5$. We can check that there are only two possible $\beta(G_{10})$-sets: $\{a, g, c, i, e\}$ and $\{f, b, h, d, j\}$. Thus, the $\gamma_t(G_{10})$-set $\{b, g, d, i\}$ intersects these two $\beta(G_{10})$-sets, and as a consequence, $\gamma_{tt}(G_{10}) = 4$. On the other hand, for the case of complete graphs K_n, $\beta(K_n) = 1$ and each vertex forms a $\beta(K_n)$-set. Thus, every vertex of K_n belongs to every $\gamma_{tt}(G)$-set and so, $\gamma_{tt}(K_n) = n$. Some other values of the independence transversal total domination number of several simple families of graphs can be straightforwardly observed. For instance, the star graph $S_{1,n-1}$ satisfies $\gamma_{tt}(S_{1,n-1}) = 2$, for the complete bipartite graph $K_{r,s}$, $\gamma_{tt}(K_{r,s}) = 2$, and for the hypercube Q_3 it follows $\gamma_{tt}(Q_3) = 4$.

Since the total domination number is not defined for graphs having isolated vertices, all the graphs considered herein have not isolated vertices. Moreover, we notice that if $H_1, H_2, \ldots, H_r$ with $r \geq 2$, are the subgraphs induced by the connected components of a graph H, then any ITTD set of minimum cardinality in H is formed by an ITTD set in one subgraph H_j and total dominating sets in the rest of subgraphs different from H_j. Thus, the following result for the case of non connected graphs is obtained.

Remark 1. Let $H_1, H_2, \ldots, H_r$ with $r \geq 2$, be the subgraphs induced by the connected components of a graph H. Then

$$\gamma_{tt}(H) = \min_{1\leq i,j\leq r} \left\{ \gamma_{tt}(H_j) + \sum_{i=1, i\neq j}^{r} \gamma_t(H_i) \right\}.$$

Proof. Let S_i and D_i be a $\gamma_{tt}(H_i)$-set and a $\gamma_t(H_i)$-set, respectively, for any $i \in \{1, \ldots, r\}$. Let $j \in \{1, \ldots, r\}$. Since S_j intersects every maximum independent set of H_j, it also intersects every $\beta(H)$-set. Moreover, $S = S_j \cup \left(\bigcup_{i=1, i\neq j}^{r} D_i\right)$ is a total dominating set of H. Thus, S is an ITTD set of H and we have

$$\gamma_{tt}(H) \leq |S| = |S_j| + \sum_{i=1, i\neq j}^{r} |D_i| \leq \min_{1\leq i,j\leq r} \left\{ \gamma_{tt}(H_j) + \sum_{i=1, i\neq j}^{r} \gamma_t(H_i) \right\}.$$

On the other hand, let A be $\gamma_{tt}(H)$-set and let $A_i = A \cap V(H_i)$ for every $i \in \{1, \ldots, r\}$. We notice that every A_i must be a total dominating set of H_i. Moreover, there must exist at least a set A_j which is an ITTD set of H_j, otherwise A would not intersect every $\beta(H)$-set. In this sense, we obtain

$$\begin{aligned}\gamma_{tt}(H) = |A| = |A_j| + \sum_{i=1, i\neq j}^{r} |A_i| \\ \geq \gamma_{tt}(H_j) + \sum_{i=1, i\neq j}^{r} \gamma_t(H_i) \\ \geq \min_{1\leq i,j\leq r} \left\{ \gamma_{tt}(H_j) + \sum_{i=1, i\neq j}^{r} \gamma_t(H_i) \right\},\end{aligned}$$

which completes the proof. □

As a consequence of the result above, from now on, we only consider the study of ITTD sets for connected graphs. So, we omit to refer to that fact throughout all our exposition.

Please cite this article in press as: A. Cabrera Martínez, et al., On the independence transversal total domination number of graphs, Discrete Applied Mathematics (2016), http://dx.doi.org/10.1016/j.dam.2016.10.033