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Discrete Applied Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Sequence mixed graphs

C. Dalfó [a,*], M.A. Fiol [b], N. López [c]

[a] Dept. de Matemàtiques, Universitat Politècnica de Catalunya, Barcelona, Catalonia, Spain

[b] Dept. de Matemàtiques, Barcelona Graduate School of Mathematics, Universitat Politècnica de Catalunya, Barcelona, Catalonia, Spain

[c] Dept. de Matemàtica, Universitat de Lleida, Lleida, Spain

ARTICLE INFO

Article history:
Received 24 May 2016
Received in revised form 27 October 2016
Accepted 28 October 2016
Available online xxxx

Keywords:
Mixed graph
Sequence graph
Line digraph
Degree/diameter problem
Moore bound
Diameter
Mean distance

ABSTRACT

A mixed graph can be seen as a type of digraph containing some edges (or two opposite arcs). Here we introduce the concept of sequence mixed graphs, which is a generalization of both sequence graphs and iterated line digraphs. These structures are proven to be useful in the problem of constructing dense graphs or digraphs, and this is related to the degree/diameter problem. Thus, our generalized approach gives rise to graphs that have also good ratio order/diameter. Moreover, we propose a general method for obtaining a sequence mixed digraph by identifying some vertices of a certain iterated line digraph. As a consequence, some results about distance-related parameters (mainly, the diameter and the average distance) of sequence mixed graphs are presented.

1. Introduction

Two techniques that have proved very useful to obtain large graphs and digraphs are, respectively, sequence graph and line digraph approaches. Sequence graphs were first proposed by Fiol, Yebra, and Fàbrega [4], whereas, within this context, line digraphs were studied by Fiol, Yebra, and Alegre [5,6]. Mixed graphs can be seen as a generalization of both, undirected and directed graphs, see for instance the works by Nguyen and Miller [11] and Buset, El Amiri, Erskine, Miller and Pérez-Rosés [3]. Here we introduce the concept of sequence mixed graphs, which is a generalization of both sequence graphs and iterated line digraphs. In particular, we show that a sequence mixed graph can be obtained by identifying some vertices of a certain iterated line digraph. This allows to apply known results of the latter to study some basic distance-related properties in the introduced structures. First, we begin with some standard definitions.

A *mixed* (or *partially directed*) graph G with vertex set V may contain (undirected) *edges* as well as directed edges (also known as *arcs*). From this point of view, a *graph* [resp. *directed graph* or *digraph*] has all its edges undirected [resp. directed]. In fact, we can identify the mixed graph G with its associated digraph G^* obtained by replacing all the edges by digons (two opposite arcs or a directed 2-cycle). The *undirected degree* of a vertex v, denoted by $d(v)$ is the number of edges incident to v. The *out-degree* [resp. *in-degree*] of vertex v, denoted by $d^+(v)$ [resp. $d^-(v)$], is the number of arcs emanating from [resp. to] v. If $d^+(v) = d^-(v) = z$ and $d(v) = r$, for all $v \in V$, then G is said to be *totally regular* of degree (r, z) (or simply (r, z)-*regular*). Note that, in this case, the corresponding digraph G^* is $(r + z)$-regular. A *walk* of length $\ell \geq 0$ from u to v is a sequence of $\ell + 1$ vertices, $u_0u_1 \ldots u_{\ell-1}u_\ell$, such that $u = u_0$, $v = u_\ell$ and each pair $u_{i-1}u_i$, for $i = 1, \ldots, \ell$, is either an edge or an arc of G. A *directed walk* is a walk containing only arcs. An *undirected walk* is a walk containing only edges. A walk

* Corresponding author.
E-mail addresses: cristina.dalfo@upc.edu (C. Dalfó), miguel.angel.fiol@upc.edu (M.A. Fiol), nlopez@matematica.udl.es (N. López).

http://dx.doi.org/10.1016/j.dam.2016.10.030
0166-218X/

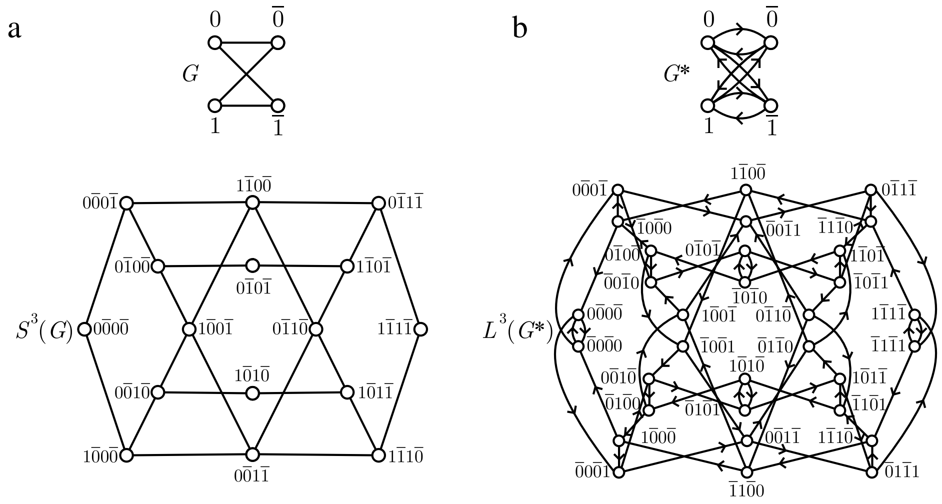

Fig. 1. (a) A graph G and its 3-sequence graph $S^3(G)$; (b) The symmetric digraph G^* and its 3-iterated line digraph $L^3(G^*)$.

whose vertices are all different is called a *path*. The length of a shortest path from u to v is the *distance* from u to v, and it is denoted by $\mathrm{dist}(u, v)$. Note that $\mathrm{dist}(u, v)$ may be different from $\mathrm{dist}(v, u)$, when shortest paths between u and v involve arcs. The maximum distance between any pair of vertices is the *diameter* k of G, while the *average distance* between vertices of G is defined as

$$\overline{k} = \frac{1}{|V|^2} \sum_{u,v \in V} \mathrm{dist}(u, v).$$

A *directed cycle* [resp. *undirected cycle*] of length ℓ is a walk of length ℓ from u to v involving only arcs [resp. edges] whose vertices are all different except $u = v$. Finally, notice that the adjacency matrix $\boldsymbol{A} = (a_{uv})$ of a mixed graph G coincides with the adjacency matrix of its associated digraph G^*, where $a_{uv} = 1$ if there is an arc from u to v, and $a_{uv} = 0$ otherwise.

The following concepts, sequence graphs, line digraphs, and related results were studied in the works by Alegre, Fàbrega, Fiol, and Yebra [4–6].

Definition 1.1. Given a digraph G, each vertex of its *ℓ-iterated line digraph* $L^\ell(G)$ represents a walk $u_0u_1 \ldots u_{\ell-1}u_\ell$ of length ℓ in G, and vertex $u = u_0u_1 \ldots u_{\ell-1}u_\ell$ is adjacent to the vertices of the form $v = u_1u_2 \ldots u_\ell u_{\ell+1}$ with $(u_\ell, u_{\ell+1})$ being an arc of G.

By way of example, in Fig. 1(b) we represent the (symmetric) digraph G^* and its 3-iterated line digraph $L^3(G^*)$. The following result shows that the line digraph technique is useful to obtain dense digraphs.

Theorem 1.2 ([5,6]). *Let G be a regular digraph different from a directed cycle, with diameter k and average distance between vertices $\overline{k}$. Then, the diameter k_ℓ and average distance $\overline{k}_\ell$ of $L^\ell(G)$ satisfy*

$$k_\ell = k + \ell, \tag{1}$$

$$\overline{k}_\ell < \overline{k} + \ell. \tag{2}$$

Moreover, if G is nonregular, then (1) also holds.

Definition 1.3. Given a graph G, the vertices of the *sequence graph* $S^\ell(G)$ of G (also known as ℓ-sequence graph) are all the walks $u_0u_1 \ldots u_{\ell-1}u_\ell$ of length ℓ of G. The edges are defined as follows: vertex $u_0u_1 \ldots u_{\ell-1}u_\ell$ is adjacent to $vu_0u_1 \ldots u_{\ell-1}$ and $u_0u_1 \ldots u_{\ell-1}w$ with (v, u_0) and $(u_{\ell-1}, w)$ being edges of G.

Within this definition, we consider a walk $u_0u_1 \ldots u_{\ell-1}u_\ell$ and its *conjugate* $u_\ell u_{\ell-1} \ldots u_1u_0$ as the same sequence (or walk). Moreover, as we only consider simple graphs, self-adjacencies (or loops) are not taken into consideration. As an example, Fig. 1(a) shows the graph $G = K_{2,2}$ and its 3-sequence graph $S^3(G)$. In our context of construction of dense graphs, the main property of sequence graphs is the following:

Theorem 1.4 ([4]). *Let G be a graph of diameter k. Then, the diameter k_ℓ of $S^\ell(G)$ satisfies*

$$k_\ell \le k + \ell.$$

Please cite this article in press as: C. Dalfó, et al., Sequence mixed graphs, Discrete Applied Mathematics (2016), http://dx.doi.org/10.1016/j.dam.2016.10.030