ARTICLE IN PRESS
Discrete Applied Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮
Contents lists available at ScienceDirect
Discrete Applied Mathematics
journal homepage: www.elsevier.com/locate/dam

Cacti with maximum eccentricity resistance-distance sum

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ARTICLE INFO

Article history:
Received 26 August 2016
Received in revised form 12 October 2016
Accepted 17 October 2016
Available online xxxx

Keywords:
Resistance distance
Eccentricity
Cactus

ABSTRACT

The eccentricity resistance-distance sum of a connected graph G is defined as $\xi^R(G) = \sum_{u,v\in V(G)}(\varepsilon(u)+\varepsilon(v))R(u,v)$, where $\varepsilon(u)$ is the eccentricity of the vertex u and $R(u,v)$ is the resistance distance between u and v in graph G. Let $Cat(n;t)$ be the set of all cacti possessing n vertices and t cycles. In this paper, some transformations of a connected graph are studied, which is mainly focused on the monotonicity on the eccentricity resistance-distance sum. By the transformation, the extremal graphs with maximum ξ^R-value of $Cat(n;t)$ are characterized.

1. Introduction

Throughout this paper, all graphs considered are simple and undirected. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. The ordinary distance $d(u, v) = d_G(u, v)$ between the vertices u and v of the graph G is the length of the shortest path between u and v, and the diameter of the graph G is defined as $d = max_{u,v\in V(G)} d(u, v)$. The eccentricity $\varepsilon_G(v)$ or $\varepsilon(v)$ of a vertex v is the distance between v and a furthest vertex from v. $d(v) = d_G(v)$ is the degree of the vertex v of the graph G. For other undefined notations and terminology from graph theory, we refer to Bollobás's book [2].

A single number that can be used to characterize some properties of the graph of a molecule is called a topological index, or graph invariant. Topological indices provide correlations with physical, chemical and thermodynamic parameters of chemical compounds [1]. The Wiener index $W(G)$ is equal to the sum of ordinary distances between all pairs of vertices, that is, $W(G) = \sum_{i<j} d(v_i, v_j)$. It is introduced in 1947 and is one of the most thoroughly studied distance-based graph invariants [7,13,25]. A modified version of the Wiener index, which is introduced by Dobrynin, Kochetova and Gutman [7], is the degree distance defined as

$$D(G) = \sum_{i<j}(d(v_i) + d(v_j))d(v_i, v_j).$$

The degree distance was widely studied [3,4,20]. Recently, a novel graph invariant called eccentric distance sum for predicting biological and physical properties was introduced by Gupta, Singh and Madan [10], which was defined as

$$\xi^d(G) = \sum_{i<j}(\varepsilon(v_i) + \varepsilon(v_j))d(v_i, v_j).$$

For more research development on the eccentric distance sum of graphs, one may be referred to [9,15,18] and the references therein. In 1993, Klein and Randić [12] introduced a distance function named resistance distance on a graph. They view a

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http://dx.doi.org/10.1016/j.dam.2016.10.032
0166-218X/

Please cite this article in press as: F. He, Z. Zhu, Cacti with maximum eccentricity resistance-distance sum, Discrete Applied Mathematics (2016), http://dx.doi.org/10.1016/j.dam.2016.10.032

graph G as an electrical network such that each edge of G is assumed to be a unit resistor, then take the resistance distance between vertices v_i and v_j to be the effective resistance between them, denoted by $R(v_i, v_j)$. The Kirchhoff index $Kf(G)$ of G is defined as $Kf(G) = \sum_{i<j} R(v_i, v_j)$, which has been widely studied [6,5,26,23,16].

In analogy with the degree distance of a graph, the degree resistance distance of a graph G was first proposed by Gutman, Feng and Yu [11] as

$$D_R(G) = \sum_{i<j}(d(v_i) + d(v_j))R(v_i, v_j).$$

Palacios [19] called this graph invariant as the additive degree-Kirchhoff index. It was systematically studied by Du, Su, Tu and Gutman in [8]. Based on the above graph invariants, Li and Wei [14] introduced a new index named the eccentricity resistance-distance sum of a graph G, which is defined as

$$\xi_R(G) = \sum_{i<j}(\varepsilon(v_i) + \varepsilon(v_j))R(v_i, v_j).$$

By the definition of $\xi_R(G)$, we have

$$\xi^R(G) = \sum_{v\in G}\varepsilon(v)\sum_{u\in G}R(v, u).$$

A cactus is a connected graph in which any two simple cycles have at most one vertex in common. Equivalently, every edge in such a graph belongs to at most one simple cycle. The set of cacti possessing n vertices and t cycles is denoted by $Cat(n; t)$, where $0 \le t \le \lfloor \frac{n-1}{2} \rfloor$. If $G \in Cat(n; t)$, then $|E(G)| = n + t - 1$. The cactus graph has many applications in real life problems, and much work has been done to study the extremal graph according to different index. Du et al. [8] determined the elements of $Cat(n; t)$ with minimum degree resistance distance are characterized. Wang and Pan [23] characterized the maximum Kirchhoff index of cacti, as well as the corresponding extremal graph. Liu et al. [17] studied the elements of $Cat(n; t)$ with second-minimum and third-minimum degree resistance distances. Li and Wei [14] determined the graph with the minimum eccentricity resistance-distance sum among $Cat(n; t)$. In this paper, we characterize the extremal graphs with maximum eccentricity resistance-distance sum among graphs in $Cat(n, t)$.

2. Preliminaries and some lemmas

Let $R_G(u, v)$ denote the resistance distance between u and v in the graph G. It is known that $R_G(u, v) = R_G(v, u)$ and $R_G(u, v) \ge 0$ with equality if and only if $u = v$. For a vertex v in G, we define $Kf_v(G) = \sum_{u\in G} R_G(u, v)$. Let $P_k = r_1r_2 \dots r_k (k \ge 2)$ be a path of G with distinct vertices $r_1, \dots, r_k$ and assume that $d(r_1) \ge 3$, $d(r_2) = \cdots = d(r_{k-1}) = 2$, then P_k is called a pendent path of G if $d(r_k) = 1$, and P_k is called an internal path if $d(r_k) \ge 3$ [24].

For the sake of brevity, in the whole of our context, for any two vertices u, v of G (or G', G''), we let $\varepsilon(v) = \varepsilon_G(v)$ (or $\varepsilon'(v) = \varepsilon_{G'}(v)$, $\varepsilon''(v) = \varepsilon_{G''}(v)$) and $R(u, v) = R_G(u, v)$ (or $R'(u, v) = R_{G'}(u, v)$, $R''(u, v) = R_{G''}(u, v)$). In the following, we give some necessary lemmas which will be used to prove our main results.

Lemma 2.1 ([11]). *Let G be a graph, x be a cut vertex of G and let u, v be vertices belonging to different components which arise upon deletion of x. Then $R_G(u, v) = R_G(u, x) + R_G(x, v)$.*

Lemma 2.2 ([12]). *Let C_k be a cycle with length k and $v \in C_k$. Then $Kf(C_k) = \frac{k^3-k}{12}$, $Kf_v(C_k) = \frac{k^2-1}{6}$.*

Lemma 2.3 ([21]). *Let G be a connected graph with a cut-vertex v such that G_1 and G_2 are two connected subgraphs of G having v as the only common vertex and $V(G_1) \cup V(G_2) = V(G)$. Let $n_1 = |V(G_1)| - 1$, $n_2 = |V(G_2)| - 1$. Then $Kf(G) = Kf(G_1) + Kf(G_2) + n_1Kf_v(G_1) + n_2Kf_v(G_2)$.*

Lemma 2.4. *Given a connected graph G with a cut vertex u and $d_G(u) \ge 3$. Let the paths $P_u = u_1u_2 \dots u_k$ and $P_v = v_1v_2 \dots v_t$ $(k \ge t)$ be the connected components of $G - u$, and let $G' = G - v_{t-1}v_t + u_kv_t$ (as shown in Fig. 1). Then $\xi^R(G) < \xi^R(G')$.*

Proof. Let $H = G - P_u - P_v$, $A = \{u_1, \dots, u_k\}$, $B = \{v_1, v_2, \dots, v_t\}$ and $C = V(H)$, and put $d = \varepsilon_H(u)$. For the transformation from G to G', $V(G) = V(G')$, we distinguish the following two cases for d.

Case 1. $d \ge t$. In this case, combining with the condition $k \ge t$, one can get that

$\varepsilon'(v_t) > \varepsilon(v_t)$; $\varepsilon'(x) \ge \varepsilon(x)$ for any $x \in V(G) \setminus v_t$; $R'(x, y) = R(x, y)$ for any $x, y \in V(G) \setminus v_t$; $R'(u_i, v_t) = k - i + 1$ and $R(u_i, v_t) = i + t$ for $i \in \{1, \dots, k\}$; $R'(v_j, v_t) = j + k + 1$ and $R(v_j, v_t) = t - j$ for $j \in \{1, \dots, t-1\}$; $R'(x, v_t) = R(x, u) + k + 1$ and $R(x, v_t) = R(x, u) + t$ for $x \in C$.

It follows that

$$\xi_1 = \sum_{x,y\in V(G)\setminus v_t}[(\varepsilon'(x) + \varepsilon'(y))R'(x, y) - (\varepsilon(x) + \varepsilon(y))R(x, y)] \ge 0.$$

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