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Implicit degree condition for hamiltonicity of 2-heavy graphs

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ABSTRACT

Let $id(v)$ denote the implicit degree of a vertex v in a graph G. An induced subgraph S of G is called f-implicit-heavy if $\max\{id(x), id(y)\} \geq |V(G)|/2$ for every pair of vertices $x, y \in V(S)$ at distance 2 in S. For a given graph R, G is called R-f-implicit-heavy if every induced subgraph of G isomorphic to R is f-implicit-heavy. For a family $\mathcal{R}$ of graphs, G is called $\mathcal{R}$-f-implicit-heavy if G is R-f-implicit-heavy for every $R \in \mathcal{R}$. G is called 2-heavy if there are at least two end-vertices of every induced claw ($K_{1,3}$) in G have degree at least $|V(G)|/2$. In this paper, we prove that: Let G be a 2-connected 2-heavy graph. If G is $\{P_7, D\}$-f-implicit-heavy or $\{P_7, H\}$-f-implicit-heavy, then G is hamiltonian.

1. Introduction

In this paper, we consider only undirected, finite and simple graphs. Notation and terminology not defined here can be found in [2]. Let G be a graph and H be a subgraph of G. For a vertex $u \in V(G)$, the *neighborhood* of u in H is denoted by $N_H(u) = \{v \in V(H) : uv \in E(G)\}$ and the *degree* of u in H is denoted by $d_H(u) = |N_H(u)|$. For two vertices $u, v \in V(H)$, the *distance* between u and v in H, denoted by $d_H(u, v)$, is the length of a shortest (u, v)-path in H. When there is no danger of ambiguity, we can use $N(u)$, $d(u)$ and $d(u, v)$ in place of $N_G(u)$, $d_G(u)$ and $d_G(u, v)$, respectively. We use $N_2(v)$ to denote the set of vertices which are at distance 2 with v, i.e. $N_2(v) = \{u \in V(G) : d(u, v) = 2\}$. Set $m_2(v) = \min\{d(u) : u \in N_2(v)\}$ and $M_2(v) = \max\{d(u) : u \in N_2(v)\}$.

An induced subgraph $K_{1,3}$ of G with vertex set $\{u, v, w, x\}$ and edge set $\{uv, uw, ux\}$ is called a *claw*, with center u and end-vertices v, w, x. For a given graph R, G is called *R-free* if G contains no induced subgraph isomorphic to R. For a family $\mathcal{R}$ of graphs, G is called *$\mathcal{R}$-free* if G is R-free for every $R \in \mathcal{R}$.

A graph G is called *hamiltonian* if it contains a *Hamilton cycle*, i.e. a cycle that contains all vertices of G. Degree condition is an important type of sufficient conditions for a graph to be hamiltonian. The following two results due to Ore and Fan, respectively, are well-known.

Theorem 1 ([10]). *Let G be a graph on $n \geq 3$ vertices. If $d(x) + d(y) \geq n$ for each pair of nonadjacent vertices x and y in G, then G is hamiltonian.*

Theorem 2 ([5]). *Let G be a 2-connected graph on $n \geq 3$ vertices. If $\max\{d(x), d(y)\} \geq n/2$ for every pair of vertices x and y at distance 2 in G, then G is hamiltonian.*

In 1989, Zhu, Li and Deng [11] found that though some vertices may have small degrees, we can use some large degree vertices to replace small degree vertices in the right position considered in the proofs, so that we may construct a longer cycle. This idea leads to the definition of implicit degree, denoted by $id(v)$ of a vertex v.

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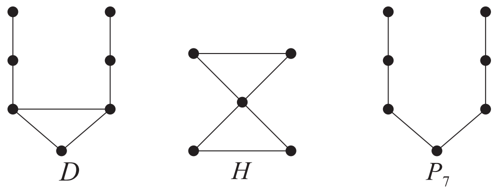

Fig. 1. Forbidden subgraph.

Definition 1 ([11]). Let v be a vertex of a graph G and $d(v) = l + 1$. If $N_2(v) \neq \emptyset$ and $d(v) \geq 2$, then let $d_1(v) \leq d_2(v) \leq d_3(v) \leq \cdots \leq d_l(v) \leq d_{l+1}(v) \leq \cdots$ be the degree sequence of vertices in $N(v) \cup N_2(v)$. Define

$$d^*(v) = \begin{cases} m_2(v), & \text{if } d_l(v) \leq m_2(v); \\ d_{l+1}(v), & \text{if } d_{l+1}(v) > M_2(v); \\ d_l(v), & \text{otherwise.} \end{cases}$$

Then the implicit degree of v is defined as $id(v) = \max\{d(v), d^*(v)\}$. If $N_2(v) = \emptyset$ or $d(v) \leq 1$, then define $id(v) = d(v)$.

Clearly, $id(v) \geq d(v)$ for every vertex v. The authors in [11] gave a sufficient condition for a 2-connected graph to be hamiltonian under implicit degree condition.

Theorem 3 ([11]). *Let G be a 2-connected graph on $n \geq 3$ vertices. If $id(x) + id(y) \geq n$ for every pair of nonadjacent vertices x and y in G, then G is hamiltonian.*

Remark 1. The introduction of implicit degree is very useful, since many classic results by considering degree conditions for the hamiltonicity of graphs can be generalized. We just give one example to show this. Fan's theorem (Theorem 2) can be easily obtained from Theorem 3. And the authors in [11] gave a simple proof to show this.

Forbidden subgraph condition is another type of sufficient conditions for hamiltonicity of graphs. The following two results are of this type, where P_7, D and H are graphs in Fig. 1.

Theorem 4 ([4]). *Let G be a 2-connected graph. If G is $\{K_{1,3}, P_7, D\}$-free, then G is hamiltonian.*

Theorem 5 ([6]). *Let G be a 2-connected graph. If G is $\{K_{1,3}, P_7, H\}$-free, then G is hamiltonian.*

A vertex v of a graph G on n vertices is called *heavy* (or *implicit-heavy*) if $d(v) \geq n/2$ (or $id(v) \geq n/2$). If v is not heavy (or not implicit-heavy), we call it *light* (or *implicit-light*). A claw of G is called *2-heavy* if at least two of its end-vertices are heavy. A subgraph S of G is called *f-heavy* (or *f-implicit-heavy*) if $\max\{d(x), d(y)\} \geq n/2$ (or $\max\{id(x), id(y)\} \geq n/2$) for every pair of vertices $x, y \in V(S)$ at distance 2 in S. For a given graph R, G is called *R-f-heavy* (or *R-f-implicit-heavy*) if every induced subgraph of G isomorphic to R is f-heavy (or *f-implicit-heavy*). For a family $\mathcal{R}$ of graphs, G is called *$\mathcal{R}$-f-heavy* (or *$\mathcal{R}$-f-implicit-heavy*) if G is R-f-heavy (or R-f-implicit-heavy) for every $R \in \mathcal{R}$. Clearly, every R-free graph is also R-f-heavy, and every R-f-heavy graph is also R-f-implicit-heavy.

In 1997, Broersma et al. [3] generalized existing results by combining degree conditions and forbidden subgraph conditions. More precisely, they restricted Fan's condition (Theorem 2) to certain subgraphs and got the following result.

Theorem 6 ([3]). *Let G be a 2-connected graph on $n \geq 3$ vertices. If G is 2-heavy and $|N(u) \cap N(v)| \geq 2$ for every pair of vertices u, v with $d(u, v) = 2$ and $\max\{d(u), d(v)\} \leq n/2$, then G is hamiltonian.*

Ning extended Theorem 4 and Theorem 5 by relaxing forbidden subgraph conditions to conditions in which the subgraphs are allowed.

Theorem 7 ([9]). *Let G be a 2-connected graph on $n \geq 3$ vertices. If G is $\{K_{1,3}, P_7, D\}$-f-heavy or $\{K_{1,3}, P_7, H\}$-f-heavy, then G is hamiltonian.*

We use implicit degree instead of degree in Theorem 7 and get the following result.

Theorem 8. *Let G be a 2-connected 2-heavy graph. If G is $\{P_7, D\}$-f-implicit-heavy or $\{P_7, H\}$-f-implicit-heavy, then G is hamiltonian.*

Remark 2. Since $id(v) \geq d(v)$ for every vertex v, it is easy to see that every graph satisfying the condition of Theorem 7 also satisfies the condition of Theorem 8, but the converse is not true. Furthermore, the following graph is a hamiltonian graph satisfying the condition of Theorem 8, but not the condition of Theorem 7.

Let $n \geq 18$ be an even integer and $K_{\frac{n}{2}} \cup K_{\frac{n}{2}-7}$ denotes the union of two complete graphs $K_{\frac{n}{2}}$ and $K_{\frac{n}{2}-7}$. Set $V(K_{\frac{n}{2}}) = \{u_1, u_2, \ldots, u_{\frac{n}{2}}\}$ and $V(K_{\frac{n}{2}-7}) = \{v_1, v_2, \ldots, v_{\frac{n}{2}-7}\}$. We construct a graph G with $V(G) = V(K_{\frac{n}{2}} \cup K_{\frac{n}{2}-7}) \cup \{x_1, x_2, \ldots, x_7\}$

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