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Complexity of rainbow vertex connectivity problems for restricted graph classes☆

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ABSTRACT

A path in a vertex-colored graph G is *vertex rainbow* if all of its internal vertices have a distinct color. The graph G is said to be *rainbow vertex connected* if there is a vertex rainbow path between every pair of its vertices. Similarly, the graph G is *strongly rainbow vertex connected* if there is a shortest path which is vertex rainbow between every pair of its vertices. We consider the complexity of deciding if a given vertex-colored graph is rainbow or strongly rainbow vertex connected. We call these problems Rainbow Vertex Connectivity and Strong Rainbow Vertex Connectivity, respectively. We prove both problems remain NP-complete on very restricted graph classes including bipartite planar graphs of maximum degree 3, interval graphs, and k-regular graphs for $k \geq 3$. We settle precisely the complexity of both problems from the viewpoint of two width parameters: pathwidth and tree-depth. More precisely, we show both problems remain NP-complete for bounded pathwidth graphs, while being fixed-parameter tractable parameterized by tree-depth. Moreover, we show both problems are solvable in polynomial time for block graphs, while Strong Rainbow Vertex Connectivity is tractable for cactus graphs and split graphs.

1. Introduction

Krivelevich and Yuster [19] introduced the concept of rainbow vertex connectivity. A path in a vertex-colored graph G is said to be *vertex rainbow* if all of its internal vertices have a distinct color. The graph G is said to be *rainbow vertex connected* if there is a vertex rainbow path between every pair of its vertices. The minimum number of colors needed to make G rainbow vertex connected is known as the *rainbow vertex connection number*, and it is denoted by $\text{rvc}(G)$. Recall the *diameter* of a graph G, denoted by $\text{diam}(G)$, is the length of a longest shortest path in G. It is easy to see two vertices u and v are rainbow vertex connected regardless of the underlying vertex-coloring if their distance $d(u, v)$ is at most 2. Thus, we have that $\text{rvc}(G) \geq \text{diam}(G) - 1$, with equality if the diameter is 1 or 2. Similarly, an easy to see upper bound is $\text{rvc}(G) \leq n - 2$, as long as we disregard the singleton graph. In other words, complete graphs are precisely the graphs with rainbow vertex connection number 0; for all other graphs we require at least 1 color.

Li et al. [21] introduced the strong variant of rainbow vertex connectivity. We say the vertex-colored graph G is *strongly rainbow vertex connected* if there is, between every pair of vertices, a shortest path whose internal vertices have a distinct color. The minimum number of colors needed to make G strongly rainbow vertex connected is known as the *strong rainbow vertex connection number*, and it is denoted by $\text{srvc}(G)$. As each strong vertex rainbow coloring is also a rainbow vertex coloring, we have that $\text{diam}(G) - 1 \leq \text{rvc}(G) \leq \text{srvc}(G) \leq n - 2$.

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Prior to the work of Krivelevich and Yuster [19], the concept of rainbow connectivity (for edge-colored graphs) was introduced by Chartrand et al. [4] as an interesting way to strengthen the connectivity property. Indeed, the notion has proven to be useful in the domain of networking [3] and anonymous communication [10]. Rainbow coloring and connectivity problems have been subject to considerable interest and research during the past years. For additional applications, we refer the reader to the survey [22]. A comprehensive introduction is also provided by the books [23,5].

It is computationally difficult to determine either $\text{rvc}(G)$ or $\text{srvc}(G)$ for a given graph G. Indeed, through the work of Chen et al. [7,6] it is known that deciding if $\text{rvc}(G) \leq k$ is NP-complete for every $k \geq 2$. Likewise, Eiben et al. [13] showed deciding if $\text{srvc}(G) \leq k$ is NP-complete for every $k \geq 3$. In the same paper, the authors also proved that the strong rainbow vertex connection number of an n-vertex graph of bounded diameter cannot be approximated within a factor of $n^{1/2-\epsilon}$, for any $\epsilon > 0$, unless P = NP. Given such strong intractability results, it is interesting to ask whether the following problem is easier.

> Rainbow Vertex Connectivity (Rvc)
> **Instance:** A connected undirected graph $G = (V, E)$, and a vertex-coloring $\psi : V \to C$, where C is a set of colors
> **Question:** Is G rainbow vertex connected under ψ?

However, Rainbow Vertex Connectivity was shown to be NP-complete by Chen et al. [7]. Later on, Huang et al. [16] showed the problem remains NP-complete even when the input graph is a line graph. A more systematic study into the complexity of Rainbow Vertex Connectivity was performed by Uchizawa et al. [30]. They proved the problem remains NP-complete for both series–parallel graphs, and graphs of bounded diameter. In contrast, they showed the problem is in P for outerplanar graphs. Furthermore, they showed the problem is fixed-parameter tractable for the n-vertex m-edge general graph parameterized by the number of colors in the vertex-coloring. That is, they gave an algorithm running in time $O(k2^k mn)$ such that given a graph vertex-colored with k colors, it decides whether G is rainbow vertex connected.

We mention two related problems, defined on edge-colored undirected graphs. A path in an edge-colored graph H is *rainbow* if no two edges of it are colored the same. The graph H is said to be *rainbow connected* if there is a rainbow path between every pair of its vertices. Likewise, the graph H is said to be *strongly rainbow connected* if there is a shortest path which is rainbow between every pair of its vertices. Formally, the two problems are defined as follows.

> Rainbow Connectivity (Rc)
> **Instance:** A connected undirected graph $H = (V, E)$, and an edge-coloring $\zeta : E \to C$, where C is a set of colors
> **Question:** Is H rainbow connected under ζ?

> Strong Rainbow Connectivity (Src)
> **Instance:** A connected undirected graph $H = (V, E)$, and an edge-coloring $\zeta : E \to C$, where C is a set of colors
> **Question:** Is H strongly rainbow connected under ζ?

It was shown by Chakraborty et al. [3] that Rainbow Connectivity is NP-complete. Later on, the complexity of both edge variants was studied by Uchizawa et al. [30]. For instance, the authors showed both problems remain NP-complete for outerplanar graphs, and that Rainbow Connectivity is NP-complete already on graphs of diameter 2. A further study into the complexity of the edge variant problems was done in our earlier work [20]. For instance, it was shown that both problems remain NP-complete on interval outerplanar graphs, k-regular graphs for $k \geq 3$, and on graphs of bounded pathwidth. In addition, block graphs were identified as a class for which the complexity of the two problems Rainbow Connectivity and Strong Rainbow Connectivity differ. Indeed, it was shown that for block graphs, Rainbow Connectivity is NP-complete, while Strong Rainbow Connectivity is in P.

In this paper, we introduce as a natural variant of Rainbow Vertex Connectivity the following problem.

> Strong Rainbow Vertex Connectivity (Srvc)
> **Instance:** A connected undirected graph $G = (V, E)$, and a vertex-coloring $\psi : V \to C$, where C is a set of colors
> **Question:** Is G strongly rainbow vertex connected under ψ?

We present several new complexity results for both Rainbow Vertex Connectivity and Strong Rainbow Vertex Connectivity.

- In Section 3, we focus on negative results. In particular, we prove both problems remain NP-complete for bipartite planar graphs of maximum degree 3 (Section 3.2), interval graphs (Section 3.3), triangle-free cubic graphs (Section 3.4), and k-regular graphs for $k \geq 4$ (Section 3.5).

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