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Two complexity results for the vertex coloring problem[☆]

D.S. Malyshev[a,*], O.O. Lobanova[b]

[a] National Research University Higher School of Economics, 25/12 Bolshaya Pecherskaya Ulitsa, 603155, Nizhny Novgorod, Russia

[b] Lobachevsky State University of Nizhny Novgorod, 23 Gagarina Avenue, Nizhny Novgorod, 603950, Russia

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ABSTRACT

We show that the chromatic number of $\{P_5, K_p - e\}$-free graphs can be computed in polynomial time for each fixed p. Additionally, we prove polynomial-time solvability of the weighted vertex coloring problem for $\{P_5, \overline{P_3 + P_2}\}$-free graphs.

1. Introduction

In this paper, we consider only *simple graphs*, i.e. finite undirected graphs without loops and multiple edges. A *coloring of a graph G* is an arbitrary mapping $c : V(G) \longrightarrow \mathbb{N}$, such that $c(u) \neq c(v)$ for any two adjacent vertices u and v of G. Elements of the set $\bigcup_{v \in V(G)} \{c(v)\}$ are said to be *colors*. A coloring c^* of a graph G is a *k-coloring* if $c^* : V(G) \longrightarrow \{1, \ldots, k\}$. The *chromatic number of a graph G*, denoted by $\chi(G)$, is the minimal number k, such that G has a k-coloring. For a given graph G and a number k, the *coloring problem* is to decide whether $\chi(G) \leq k$ or not. A similar *k-colorability problem* is to check whether vertices of a given graph can be colored with at most k colors. Both problems can be naturally defined in another way via partition into independent sets. An *independent set of a graph* is an arbitrary set of its pairwise non-adjacent vertices. A graph coloring is a partition of vertex set of a given graph into independent sets, called *color classes*.

For a given graph G and a function $w : V(G) \longrightarrow \mathbb{N}$, a pair (G, w) is called a *weighted graph*. For a weighted graph (G, w), the *weighted coloring problem* is to find the smallest number k, denoted by $\chi_w(G)$, such that there is a function $c : V(G) \longrightarrow 2^{\{1,2,\ldots,k\}}$, where $|c(v)| = w(v)$ for any $v \in V(G)$ and $c(v_1) \cap c(v_2) = \emptyset$ for any edge (v_1, v_2) of G. The number $\chi_w(G)$ is called the *weighted chromatic number* of (G, w). For any graph G, $\chi_{w'}(G) = \chi(G)$, where w' maps every vertex to 1. So, the weighted coloring problem generalizes the coloring problem.

A class of simple graphs is called *hereditary* if it is closed under deletion of vertices. It is well-known that any hereditary (and only hereditary) graph class $\mathcal{X}$ can be defined by a set of its forbidden induced subgraphs $\mathcal{S}$. We write $\mathcal{X} = Free(\mathcal{S})$, and the graphs in $\mathcal{X}$ are said to be $\mathcal{S}$*-free*. If $\mathcal{S} = \{G\}$, then we write "G-free" instead of "$\{G\}$-free".

There is a natural lower bound for the chromatic number of graphs. A *clique* in a graph is a subset of its pairwise adjacent vertices. The size of a maximum clique of a graph G, denoted by $\omega(G)$, is called the *clique number* of G. Clearly, $\chi(G) \geq \omega(G)$.

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* Corresponding author.
E-mail addresses: dsmalyshev@rambler.ru (D.S. Malyshev), olga-olegov@yandex.ru (O.O. Lobanova).

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A graph is said to be *perfect* if the clique and the chromatic numbers are equal for its every induced subgraph, not necessarily proper. The class of perfect graphs coincides with $Free(\{C_5, \overline{C_5}, C_7, \overline{C_7}, \ldots\})$, by The Strong Perfect Graph Theorem [5], see notation for graphs in the next section. Sometimes, computing the clique number in polynomial time helps to determine the chromatic number also in polynomial time [15,36]. More precisely, for graphs in [15,36], including perfect graphs, determining the chromatic number can be polynomially reduced to computing the clique number and the clique number can be found in polynomial time.

The computational complexity of the coloring, the weighted coloring, and the k-colorability problems and their edge variants was intensively studied for families of the forms $\{Free(\mathcal{S})|\ \mathcal{S}$ has a small number of graphs$\}$ and $\{Free(\mathcal{S})|$ every graph in $\mathcal{S}$ is small$\}$ [1–3,7,8,13,14,17–39,41,42]. The computational complexity of the coloring problem was completely determined for all the classes of the form $Free(\{G\})$ [22]. Namely, if $\subseteq_i$ is the induced subgraph relation, then the problem is polynomial-time solvable for $Free(\{G\})$ whenever $G \subseteq_i P_4$ or $G \subseteq_i P_3 + K_1$; otherwise it is NP-complete. A study of forbidden pairs was also initiated in [22].

The following result shows some recent advances in classification of the complexity of the coloring problem for $\{G_1, G_2\}$-free graphs [12]. Note that, by symmetry, the graphs G_1 and G_2 may be swapped in each of the subcases of the theorem.

Theorem 1. *Let G_1 and G_2 be two fixed graphs. The coloring problem is* NP-*complete for $Free(\{G_1, G_2\})$ if:*

1. *$C_p \subseteq_i G_1$ for $p \geq 3$, and $C_q \subseteq_i G_2$ for $q \geq 3$*
2. *$K_{1,3} \subseteq_i G_1$, and $K_{1,3} \subseteq_i G_2$ or $\overline{K_2 + O_2} \subseteq_i G_2$ or $C_r \subseteq_i G_2$ for $r \geq 4$ or $K_4 \subseteq_i G_2$*
3. *G_1 and G_2 contain a spanning subgraph of a $2K_2$ as an induced subgraph*
4. *bull $\subseteq_i G_1$, and $K_{1,4} \subseteq_i G_2$ or $\overline{C_4 + K_1} \subseteq_i G_2$*
5. *$C_3 \subseteq_i G_1$, and $K_{1,p} \subseteq_i G_2$ for $p \geq 5$*
6. *$C_3 \subseteq_i G_1$ and $P_{22} \subseteq_i G_2$*
7. *$C_p \subseteq_i G_1$ for $p \geq 5$, and G_2 contains a spanning subgraph of a $2K_2$ as an induced subgraph*
8. *$C_p + K_1 \subseteq_i G_1$ for $p \in \{3, 4\}$ or $\overline{C_q} \subseteq_i G_1$ for $q \geq 6$, and G_2 contains a spanning subgraph of a $2K_2$ as an induced subgraph*
9. *$K_5 \subseteq_i G_1$ and $P_7 \subseteq_i G_2$*
10. *$K_6 \subseteq_i G_1$ and $P_6 \subseteq_i G_2$.*

It is polynomial-time solvable for $Free(\{G_1, G_2\})$ if:

1. *G_1 is an induced subgraph of a P_4 or a $P_3 + K_1$*
2. *$G_1 \subseteq_i K_{1,3}$, and $G_2 \subseteq_i$ hammer or $G_2 \subseteq_i$ bull or $G_2 \subseteq_i P_5$*
3. *$G_1 \neq K_{1,5}$ is a forest on at most six vertices or $G_1 = K_{1,3} + 3K_1$, and $G_2 \subseteq_i$ paw*
4. *$G_1 \subseteq_i sK_2$ or $G_1 \subseteq_i P_5 + O_s$ for $s > 0$, and G_2 is a complete graph or $G_2 \subseteq_i$ hammer*
5. *$G_1 \subseteq_i P_4 + K_1$ or $G_1 \subseteq_i P_5$, and $G_2 \subseteq_i \overline{P_4 + K_1}$ or $G_2 \subseteq_i \overline{P_5}$*
6. *$G_1 \subseteq_i K_2 + O_2$, and $G_2 \subseteq_i \overline{2K_2 + K_1}$ or $G_2 \subseteq_i \overline{P_3 + O_2}$ or $G_2 \subseteq_i \overline{P_3 + K_2}$*
7. *$G_1 \subseteq_i \overline{K_2 + O_2}$, and $G_2 \subseteq_i 2K_2 + K_1$ or $G_2 \subseteq_i P_3 + O_2$ or $G_2 \subseteq_i P_3 + P_2$*
8. *$G_1 \subseteq_i K_2 + O_s$ for $s > 0$ or $G_1 = P_5$, and $G_2 \subseteq_i \overline{K_2 + O_t}$ for $t > 0$*
9. *$G_1 \subseteq O_4$ and $G_2 \subseteq \overline{P_3 + O_2}$*
10. *$G_1 \subseteq P_5$, and $G_2 \subseteq_i C_4$ or $G_2 \subseteq \overline{P_3 + O_2}$.*

A complete complexity dichotomy for the coloring problem is hard to obtain even in the following cases: (a) two forbidden induced subgraphs, each on at most four vertices [24]; (b) two connected forbidden induced subgraphs, each on at most five vertices [32]. For all but three cases either NP-completeness or polynomial-time solvability was shown in the family of all the hereditary classes, defined by four-vertex forbidden induced structures [24]. The remaining three classes $Free(\{O_4, C_4\})$, $Free(\{K_{1,3}, O_4\})$, $Free(\{K_{1,3}, O_4, K_2+O_2\})$ are stubborn. A similar result was obtained in [32] for two connected five-vertex forbidden induced fragments, where the number of open cases was 13. A list of the open cases is presented below (the numbers in parentheses show the quantities of such kind sets).

1. $\{K_{1,3}, G\}$, where $G \in \{$*bull*, *butterfly*$\}$ (2)
2. $\{$*fork*, *bull*$\}$ (1)
3. $\{P_5, G\}$, where G is an arbitrary connected five-vertex complement graph of the line graph of a forest with 3 leaves in each connected component and $G \notin \{K_5, gem\}$ (10).

Recently, the number of the open cases was reduced to 10 [18,36] by proving that the coloring problem can be solved in polynomial time for $Free(\{P_5, \overline{P_5}\})$, $Free(\{K_{1,3}, bull\})$, $Free(\{P_5, \overline{P_3 + O_2}\})$. In this paper, we reduce the number to eight by showing that the coloring problem can be solved for $\{P_5, \overline{P_3 + P_2}\}$-free or $\{P_5, K_p - e\}$-free graphs in polynomial time. More generally, we prove polynomial-time solvability of the weighted coloring problem for $\{P_5, \overline{P_3 + P_2}\}$-free graphs.

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