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Nordhaus–Gaddum-type results for the Steiner Wiener index of graphs☆

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ABSTRACT

The Wiener index W of a connected graph G with vertex set $V(G)$ is defined as $W = \sum_{u,v\in V(G)} d(u, v)$ where $d(u, v)$ stands for the distance between the vertices u and v of G. For $S \subseteq V(G)$, the Steiner distance $d(S)$ of the vertices of S is the minimum size of a connected subgraph of G whose vertex set contains S. The kth Steiner Wiener index $SW_k(G)$ of G is defined as the sum of Steiner distances of all k-element subsets of $V(G)$. In 2005, Zhang and Wu studied the Nordhaus–Gaddum problem for the Wiener index. We now obtain analogous results for SW_k, namely sharp upper and lower bounds for $SW_k(G)+SW_k(\overline{G})$ and $SW_k(G)\cdot SW_k(\overline{G})$, valid for any connected graph G whose complement $\overline{G}$ is also connected.

1. Introduction

All graphs in this paper are assumed to be undirected, finite and simple and connected. We refer to [4] for graph theoretical notation and terminology not specified here. For a graph G, let $V(G)$, $E(G)$ and $e(G) = |E(G)|$ denote the set of vertices, the set of edges and the size of G, respectively.

If S is a vertex-subset of a graph G, the subgraph of G induced by S is denoted by $G[S]$. We denote by $E_G[X, Y]$ the set of edges of G with one end in X and the other in Y. If $X = \{x\}$, we simply write $E_G[x, Y]$ for $E_G[\{x\}, Y]$.

The *connectivity* of a graph G, written $\kappa(G)$, is the order of a minimum vertex-subset $S \subseteq V(G)$ such that $G - S$ is disconnected or has only one vertex. Thus, if G is connected, then $\kappa(G) \geq 1$; if G has cut vertices, then $\kappa(G) = 1$.

The introduction is divided into the three subsections, in order to state the motivations and results of this paper.

1.1. Distance and its generalization

Distance is one of the basic concepts of graph theory [5]. If G is a connected graph and $u, v \in V(G)$, then the *distance* $d(u, v)$ between u and v is the length of a shortest path connecting u and v.

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The distance between two vertices u and v in a connected graph G also equals the minimum size of a connected subgraph of G containing both u and v. This observation suggests a generalization of the distance concept. The *Steiner distance* of a graph, introduced by Chartrand et al. in 1989 [7], is a natural and consequent generalization of the classical graph distance. For a graph $G(V,E)$ and a set $S \subseteq V(G)$ of at least two vertices, *an S-Steiner tree* or *a Steiner tree connecting S* (or simply, *an S-tree*) is a subgraph $T(V',E')$ of G that is a tree with $S \subseteq V'$. Then the *Steiner distance* $d_G(S)$ of the vertices of S (or simply the distance of S) is the minimum size of all connected subgraphs whose vertex sets contain S. Observe that $d_G(S) = \min\{e(T) \mid S \subseteq V(T)\}$, where T is a subtree of G. Furthermore, if $S = \{u, v\}$, then $d_G(S)$ coincides with the classical distance between u and v.

Observation 1. *Let G be a connected graph of order n and k be an integer, $2 \leq k \leq n$. If $S \subseteq V(G)$ and $|S| = k$, then $k-1 \leq d_G(S) \leq n-1$.*

The *average Steiner distance* $\mu_k(G)$ of a graph G, introduced by Dankelmann et al. [8,9], is defined as the average of the Steiner distances of all k-subsets of $V(G)$, i.e.,

$$\mu_k(G) = \binom{n}{k}^{-1} \sum_{\substack{S\subseteq V(G)\\ |S|=k}} d_G(S). \tag{1}$$

Let n and k be integers such that $2 \leq k \leq n$. The *Steiner k-eccentricity* $e_k(v)$ of a vertex v of G is defined by $e_k(v) = \max\{d(S) \mid S \subseteq V(G), |S| = k, v \in S\}$. The *Steiner k-radius* of G is $srad_k(G) = \min\{e_k(v) \mid v \in V(G)\}$, whereas the *Steiner k-diameter* of G is $sdiam_k(G) = \max\{e_k(v) \mid v \in V(G)\}$. Note that for any vertex v of any connected graph G, $e_2(v) = e(v)$, and in addition $srad_2(G) = rad(G)$ and $sdiam_2(G) = diam(G)$. For more details on Steiner distance we refer to [2,6–9,12,18,20].

Mao [18] obtained the following results. By $\Delta(G)$ we denote the maximum degree among all vertices of G.

Lemma 2 ([18]). *Let G be a connected graph with connected complement $\overline{G}$. If $sdiam_k(G) \geq 2k$, then $sdiam_k(\overline{G}) \leq k$.*

Lemma 3 ([18]). *Let G be a connected graph of order n. Then $sdiam_3(G) = 2$ if and only if $0 \leq \Delta(\overline{G}) \leq 1$.*

Lemma 4 ([18]). *Let n, k be integers such that $2 \leq k \leq n$, and let G be a connected graph of order n. If $sdiam_k(G) = k-1$, then $0 \leq \Delta(\overline{G}) \leq k-2$.*

Lemma 5 ([18]). *Let G be a connected graph of order n with connected complement. Let k be an integer such that $3 \leq k \leq n$. Let $x = 0$ if $n \geq 2k-2$ and $x = 1$ if $n < 2k-2$. Then*

(1) $2k-1-x \leq sdiam_k(G) + sdiam_k(\overline{G}) \leq \max\{n+k-1, 4k-2\}$;
(2) $(k-1)(k-x) \leq sdiam_k(G) \cdot sdiam_k(\overline{G}) \leq \max\{k(n-1), (2k-1)^2\}$.

Lemma 6 ([18]). *Let G be a graph. Then $sdiam_{n-1}(G) = n-2$ if and only if G is 2-connected.*

The following corollary is immediate from the above lemmas.

Corollary 7. *Let G and $\overline{G}$ be connected graphs. If $sdiam_3(G) \geq 6$, then $sdiam_3(\overline{G}) = 3$.*

Proof. By Lemma 2, $sdiam_3(\overline{G}) \leq 3$. We claim that $sdiam_3(\overline{G}) = 3$. Assume, to the contrary, that $sdiam_3(\overline{G}) = 2$. By Lemma 3, we would have $0 \leq \Delta(G) \leq 1$, implying that G is not connected. If so, then the requirement $sdiam_3(G) \geq 6$ cannot hold, i.e., it cannot be $sdiam_3(\overline{G}) = 2$. □

1.2. Wiener index and its generalization

The *Wiener index* is defined as the sum of ordinary distances of all pairs of vertices of the underlying graph, i.e., as

$$W(G) = \sum_{u,v\in V(G)} d(u,v)$$

and its mathematical theory is nowadays well elaborated. For details see the surveys [11,22], the recent papers [10,21,13,15] and the references cited therein.

Li et al. [17] generalized the concept of Wiener index using Steiner distance, by defining the *Steiner k-Wiener index* $SW_k(G)$ of the connected graph G as

$$SW_k(G) = \sum_{\substack{S\subseteq V(G)\\ |S|=k}} d_G(S).$$

However, with regard to this definition, one should bear in mind Eq. (1), and the Refs. [8,9].