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The k-rainbow reinforcement numbers in graphs

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ABSTRACT

Let $k \geq 1$ be an integer, and let G be a graph. A *k-rainbow dominating function* (or a *k-RDF*) of G is a function f from the vertex set $V(G)$ to the family of all subsets of $\{1, 2, \ldots, k\}$ such that for every $v \in V(G)$ with $f(v) = \emptyset$, the condition $\bigcup_{u\in N_G(v)} f(u) = \{1, 2, \ldots, k\}$ is fulfilled, where $N_G(v)$ is the open neighborhood of v. The *weight* of a k-RDF f of G is the value $\omega(f) = \sum_{v\in V(G)} |f(v)|$. The *$k$-rainbow domination number* of G, denoted by $\gamma_{rk}(G)$, is the minimum weight of a k-RDF of G. The 1-rainbow domination is the same as the classical domination.

The *k-rainbow reinforcement number* of G, denoted by $r_{rk}(G)$, is the minimum number of edges that must be added to G in order to decrease the k-rainbow domination number. In this paper, we study the k-rainbow reinforcement number of graphs to compare γ_{rk} and $\gamma_{rk'}$ for $k \neq k'$, and present some sharp bounds concerning the invariant.

1. Introduction

In this paper, all graphs are finite, simple, and undirected. Let G be a graph. We let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. For a vertex $v \in V(G)$, the *open neighborhood* of v, denoted by $N_G(v)$, is the set $\{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of v, denoted by $N_G[v]$, is the set $N_G(v) \cup \{v\}$. For a set $U \subseteq V(G)$ and a vertex $v \in U$, the *private neighborhood* of v with respect to U, denoted by $\mathrm{pn}_G[v, U]$, is the set $\{u \in V(G) : N_G[u] \cap U = \{v\}\}$. The *degree* of $v \in V(G)$, denoted by $d_G(v)$, is defined by $d_G(v) = |N_G(v)|$. We let $\Delta(G)$ denote the *maximum degree* of G. The *complement* of G is denoted by $\overline{G}$. We let P_n and C_n denote the *path* and the *cycle* of order n, respectively. We let $K_{m_1,\ldots,m_t}$ denote the *complete t-partite graph* with t partite sets having cardinalities $m_1, \ldots, m_t$. Consult [6,12] for notation and terminology which are not defined here.

The classical domination concept in a graph represents situations in which every location that is occupied by no guard requires the presence of one guard in a neighboring location. Here we assume a more complex situation that, for example, there are different types of guards and it is required that each location which is occupied by no guard has all types of guards in its neighborhood. Brešar, Henning and Rall [2] introduced the rainbow domination concept to consider such situation. Let $k \geq 1$ be an integer, and set $[k] := \{1, 2, \ldots, k\}$. A function $f : V(G) \to 2^{[k]}$ is a *k-rainbow dominating function* (or a *k-RDF*) of G if for every $v \in V(G)$ with $f(v) = \emptyset$, the condition $\bigcup_{u\in N_G(v)} f(u) = [k]$ is fulfilled. The *weight* of a k-RDF f of G is the

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value $\omega(f) := \sum_{v\in V(G)} |f(v)|$. The *k-rainbow domination number* of G, denoted by $\gamma_{rk}(G)$, is the minimum weight of a k-RDF of G. A k-RDF f of G is a *γ_{rk}-function* if $\omega(f) = \gamma_{rk}(G)$. Note that $\gamma_{r1}(G)$ is equal to the classical domination number, denoted by $\gamma(G)$.

Brešar and Šumenjak [3] proved that the 2-rainbow domination problem is NP-complete, and Chang, Wu and Zhu [4] proved that the k-rainbow domination problem is also NP-complete for every k. Thus it is important to find good bounds on the k-rainbow domination number. For example, the following sharp upper bounds have been known.

- For every connected graph G of order $n \geq 2$, $\gamma_{r1}(G) = \gamma(G) \leq \frac{n}{2}$ [10].
- For every connected graph G of order $n \geq 3$, $\gamma_{r2}(G) \leq \frac{3n}{4}$ [13].
- For every connected graph G of order $n \geq 5$, $\gamma_{r3}(G) \leq \frac{8n}{9}$ [5].

However, when we consider γ_{rk} with $k \geq 4$, the situation is changed drastically. For a graph G, the function assigning $\{1\}$ to each vertex of G is a k-RDF of G with $\omega(f) = |V(G)|$. Thus

$$\gamma_{rk}(G) \leq n \quad \text{for every (connected) graph } G \text{ of order } n. \tag{1.1}$$

Since $\gamma_{rk}(P_n) = n$ for $k \geq 4$ and $n \geq 1$ (see [5]), the bound in (1.1) is best possible. In particular, in such type of upper bounds of the rainbow domination, we cannot distinguish γ_{rk} and $\gamma_{rk'}$ for $4 \leq k < k'$. Furthermore, for two integers k and k' with $2 \leq k < k' < 2k$, we have $\gamma_{rk}(K_{k',m}) = \gamma_{rk'}(K_{k',m})$ $(m \geq k')$. In actuality, there are many graphs G which outwardly have no difference between $\gamma_{rk}(G)$ and $\gamma_{rk'}(G)$ for $k \neq k'$. When we are faced with such a problem, one might be interested in some potential properties concerning the rainbow domination of G to distinguish the k-rainbow domination and the k'-rainbow domination. In this paper, our main aim is to formulate such a potential property of the rainbow domination.

Here we focus on the reinforcement concept for the domination. The *reinforcement number* of a graph G with $\gamma(G) \geq 2$, denoted by $r(G)$, is the minimum number of edges that must be added to G in order to decrease the domination number. The reinforcement number $r(G)$ of a graph G is defined to be 0 if $\gamma(G) = 1$. The reinforcement number was introduced by Kok and Mynhardt [9] and has been studied in, for example, [1,7,8].

We extend the reinforcement number to the rainbow domination. Let $k \geq 1$ be an integer. For a graph G, a subset F of $E(\overline{G})$ is a *k-rainbow reinforcement set* (or a *k-RRS*) of G if $\gamma_{rk}(G+F) < \gamma_{rk}(G)$. Note that G has a k-RRS if and only if $\gamma_{rk}(G) \geq k+1$. The *k-rainbow reinforcement number* of a graph G with $\gamma_{rk}(G) \geq k+1$, denoted by $r_{rk}(G)$, is the minimum size of a k-RRS of G. The k-rainbow reinforcement number $r_{rk}(G)$ of a graph G is defined to be 0 if $\gamma_{rk}(G) \leq k$. A k-RRS F of G is an *r_{rk}-set* if $|F| = r_{rk}(G)$. Note that $r_{r1}(G)$ is equal to the original reinforcement number $r(G)$.

Let G be a graph with $\gamma_{rk}(G) = \gamma_{rk'}(G)$ for $k \neq k'$. Recall that our aim is to formulate a potential property concerning the rainbow domination of G. We will show that there is a magnitude relation between $r_{rk}(G)$ and $r_{rk'}(G)$ (see Theorem 3.1 in Section 3), and hence k-rainbow reinforcement number is an index that we desire.

Our purpose in this paper is to initiate the study of k-rainbow reinforcement number in graphs. In Section 2, we study the k-rainbow reinforcement number under some special conditions. Recall that our first motivation is to distinguish the k-rainbow domination and the k'-rainbow domination of a graph G with $\gamma_{rk}(G) = \gamma_{rk'}(G)$. In Section 3, we consider such a situation. In Section 4, we present an upper bound of r_{rk} for general graphs. In general, for a given graph G, it is difficult to determine $r_{rk}(G)$. In Section 5, we focus on the 2-rainbow reinforcement number, and determine exact value of r_{r2} for some classes of graphs, and give a sharp upper bound of r_{r2} for trees.

1.1. Preliminaries

In this subsection, we give further definitions and present two useful lemmas.

Let G be a graph, and let f be a k-RDF of G. For $I \subseteq [k]$, we let $V_I^f(G) = \{v \in V(G) : f(v) = I\}$ and $U_I^f(G) = \{v \in V(G) : f(v) \supseteq I\}$. When the considered graph G is clear, we omit the symbol "(G)"; when the elements of a set I are specified, we usually use a sequence of elements of I instead of I. Thus, for example, we use V_1^f, $V_{1,2}^f$ and $U_{1,2}^f$ instead of $V_{\{1\}}^f(G)$, $V_{\{1,2\}}^f(G)$ and $U_{\{1,2\}}^f(G)$, respectively. Furthermore, we use $V_0^f(G)$ or V_0^f instead of $V_\emptyset^f(G)$.

The following fundamental lemmas will be used in our proof.

Lemma 1.1 ([2]). *Let $k \geq 1$ be an integer, and let G be a graph. Then $\gamma_{rk}(G) \leq \gamma_{r(k+1)}(G)$.*

Lemma 1.2. *Let $k \geq 1$ be an integer, and let G be a graph with $\gamma_{rk}(G) \geq k+1$. Let F be an r_{rk}-set of G, and let g be a γ_{rk}-function of $G+F$. Then the following hold:*

(i) *For each edge $v_1v_2 \in F$, there is an integer $i \in \{1, 2\}$ such that $g(v_i) = \emptyset$ and $g(v_{3-i}) \neq \emptyset$.*
(ii) *We have $\gamma_{rk}(G+F) = \gamma_{rk}(G) - 1$.*

Proof. We first show (i). If there exists an edge $v_1v_2 \in F$ such that either $g(v_i) \neq \emptyset$ for each $i \in \{1, 2\}$ or $g(v_1) = g(v_2) = \emptyset$, then g is also a k-RDF of $G+(F-\{v_1v_2\})$, and hence $F-\{v_1v_2\}$ is a k-RRS of G, which contradicts the definition of F. Therefore (i) holds.