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Scaling laws for maximum coloring of random geometric graphs

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ABSTRACT

We examine maximum vertex coloring of random geometric graphs, in an arbitrary but fixed dimension, with a constant number of colors. Since this problem is neither scale-invariant nor smooth, the usual methodology to obtain limit laws cannot be applied. We therefore leverage different concepts based on subadditivity to establish convergence laws for the maximum number of vertices that can be colored. For the constants that appear in these results, we provide the exact value in dimension one, and upper and lower bounds in higher dimensions.

1. Introduction

We examine maximum coloring of random geometric graphs (RGGs), in an arbitrary but fixed dimension d, with a constant number of colors. The vertices of an RGG (whose spatial distribution will be defined below) are embedded in an Euclidean space that is equipped with the ℓ_2 distance or some ℓ_p distance in general, and two vertices are connected if and only if they are within a given Euclidean distance r. More specifically, we address the questions: *What is the maximum number of vertices in a sparse RGG that can be properly colored with a constant number of colors? In particular, what is the asymptotic behavior of that value, as the total number of vertices in the graph tends to infinity?*

It is important to emphasize the distinction between our problem and that of determining the chromatic number, which is the minimum required number of colors to properly color all the vertices of a graph such that no two adjacent vertices are assigned the same color. Determining whether an RGG (or a unit-disk graph) is k-colorable, i.e., whether its chromatic number is at most k, is NP-hard even for $k = 3$, see [3]. Our problem is different from determining the chromatic number, since we are interested in the maximum number of vertices that can be properly colored with given $k \in \mathbb{N}$ colors, as well as from k-colorability, which is a binary decision problem. The chromatic number of RGGs has been studied in detail (for different values of the expected degree), see Theorem 1.1 in [8]. The chromatic number in the *thermodynamic regime*, when the expected degree is constant, is 'almost' logarithmic in the number of vertices n, i.e., $(1 + o(1)) \log n / \log \log n$, which additionally inspires our problem where only a constant number of colors is available.

The above-mentioned questions are not only of fundamental interest, but also motivated by applications in wireless networks, where the various users need to be assigned channels (transmission frequencies) in order to be able to communicate, subject to certain interference constraints. For example, in order to avoid excessive interference, the same channel cannot be assigned to two users within a certain reuse distance r. The total number of required channels to cover all

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users then corresponds to the chromatic number of the associated interference graph where two users are neighbors when they are located within distance r. When the user locations are governed by a spatial Poisson process, the interference graph is an RGG, and the chromatic number will grow without bound as the total number of users grows large. As a result, the required number of channels to cover all users will grow without bound, implying that the capacity per channel, and hence the so-called max–min throughput of the network, will vanish in the limit, which is obviously undesirable. The question thus arises how many users can be covered when the number of available channels is finite. It will then not be feasible to cover all users as the total number of users grows large, but the users that do get covered are ensured to receive a strictly positive throughput. The results that we prove in the present paper imply that any target for the fraction of users to be covered, arbitrarily close to one, can be achieved in the limit with a sufficiently large but constant number of channels.

Besides wireless networks, RGGs have also found applications in various further areas, e.g. cluster analysis, statistical physics, modeling data in high-dimensional spaces, and hypothesis testing, to mention just a few [10]. For problems on many of these 'real' networks, the sparse regime with constant expected vertex degree is particularly relevant, see [7].

We now formally state the main problem and results. For any subset of points $V \subseteq \mathbb{R}^d$ and $r \in \mathbb{R}_+$, let $G_r(V)$ be the graph with vertex set V and edge set $E = \{\{u, v\} \in V^2 : \|u - v\| \leq r\}$, i.e., connecting all pairs of points that are within a given Euclidean distance r.

The main object of interest is the cardinality of a set obtained by a maximum proper coloring with k colors of a given graph $G_r(V)$. Note that such a set obtained by a maximum proper coloring on finite V may not be unique, but its cardinality is unique and defined as follows. For any $k \in \mathbb{N}$, let $N_{k,r}(V)$ be the maximum number of vertices that can be properly colored in $G_r(V)$ with k colors. For any $\lambda > 0$, let $\mathcal{X}_\lambda$ be a Poisson point process of intensity λ in $\mathbb{R}^d$. For compactness, denote

$$F_{k,\lambda}(t) = N_{k,1}([0, t]^d \cap \mathcal{X}_\lambda)$$

for any $t \geq 0$.

Also, let $\mathcal{I}_n$ be a collection of n points uniformly and independently distributed in the unit cube $[0, 1]^d$. For compactness, denote

$$H_{k,r}(n) = N_{k,r}(\mathcal{I}_n)$$

for any $n \in \mathbb{N}$ and $r > 0$.

The main problem: We are interested in the asymptotic behavior of the expectation and moreover the distribution of $F_{k,\lambda}(t)$ as $t \to \infty$, as well as $H_{k,\nu}(n)$ as $n \to \infty$.

The main results: We show that for any $d, k \in \mathbb{N}$ and $\lambda > 0$, the functional $F_{k,\lambda}(t)$ converges in probability

$$\frac{F_{k,\lambda}(t)}{\lambda t^d} \xrightarrow{\text{p}} a_{k,\lambda},$$

for some $a_{k,\lambda} \in (0, 1]$, and in distribution, for any $\nu > 0$,

$$\frac{H_{k,\sqrt[d]{\nu/n}}(n)}{n} \xrightarrow{\text{d}} a_{k,\nu}.$$

One of our main methods involves the notion of *subadditivity*. Concretely, we divide the cube $[0, t]^d$ into cubes of volume s^d, for some $s < t$ which we specify later, and apply the subadditivity argument in order to relate $F_{k,\lambda}(t)$ and $F_{k,\lambda}(s)$. We show that the lower and upper limits as $t \to \infty$ of $F_{k,\lambda}(t)$ exist and are the same, and moreover we establish the weak law of large numbers for $F_{k,\lambda}(t)$ and the strong law of large numbers for $H_{k,\nu}(n)$.

In Lemma 4, we prove that the variance $\mathbb{V}ar\left\{F_{k,\lambda}(t)\right\} = \Omega(t^d)$, i.e., the limiting variance normalized by t^d is bounded away from 0, and in Lemma 6 we present an upper bound on $\mathbb{V}ar\left\{F_{k,\lambda}(t)\right\} = O(t^d)$, which together imply $\mathbb{V}ar\left\{F_{k,\lambda}(t)\right\} = \Theta(t^d)$, see Lemma 7.

There are two branches of methods prevalent in discrete stochastic geometry, subadditive and stabilization methods, usually used to obtain the limiting behavior of some Euclidean functionals: laws of large numbers, central limit theorems, etc. For an excellent survey, the reader is referred to Yukich [14].

At first glance, the results in this paper can be seen as a subproblem and amenable to analysis by using techniques from [2], and even the "more general subadditive methods" developed by Steele in Chapter 3 of [13]. In order to apply these techniques from [13], a function L that maps a finite subset of points from $\mathbb{R}^d$ to $\mathbb{R}_+$ must satisfy the following four hypotheses: (i) normalization $L(\emptyset) = 0$; (ii) homogeneity $L(\alpha x_1, \alpha x_2, \ldots, \alpha x_n) = \alpha L(x_1, x_2, \ldots, x_n)$ for every $\alpha > 0$; (iii) translation invariance $\forall y \in \mathbb{R}^d$ $L(x_1 + y, x_2 + y, \ldots, x_n + y) = L(x_1, x_2, \ldots, x_n)$; (iv) geometric subadditivity, where for all $m, n \geq 1$ and $x_1, x_2, \ldots, x_n \in [0, 1]^d$ we have

$$L(x_1, x_2, \ldots, x_n) \leq \sum_{i=1}^{m^d} L(\{x_1, x_2, \ldots, x_n\} \cap Q_i) + O(m^{d-1}), \tag{1}$$

where the unit cube $[0, 1]^d$ is partitioned into m^d cubes Q_i with side $1/m$.

Additionally, (v) L is monotone, if for all n and x_i, $L(x_1, \ldots, x_n) \leq L(x_1, \ldots, x_n, x_{n+1})$. For example, Steele proves a so-called "basic theorem", see Theorem 3.1.1 in [13], for general subadditive Euclidean functionals that are monotone and